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A new proof of standard completeness for the uninorm logic **UL***

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【Abstract】This paper investigates a new proof of standard completeness (i.e. completeness on the real unit interval [0, 1]) for the uninorm (based) logic **UL** introduced by Metcalfe and Montagna in [15]. More exactly, standard completeness is established for **UL** by using nuclear completions method introduced in [8, 9].

【Key words】(substructural) fuzzy logic, fuzzy logic, uninorm (based) logic

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1. Introduction

In this paper we investigate a new proof of standard completeness (i.e., completeness on the real unit interval [0, 1]) of the uninorm logic **UL**. For this, we first recall briefly some historical facts associated with fuzzy logic, which are mentioned in [22].

Many-valued logics with truth values in the real unit interval [0, 1] have a long and distinguished history, and the well-known examples are the infinite-valued systems **Ł** (Łukasiewicz logic), **G** (Gödel-Dummett logic), and Π (Product logic). In particular, Hájek [11] introduced **BL** (Basic fuzzy logic) and showed that **Ł**, **G**, and Π are its extensions. **BL** is the most important logic of *continuous* t-norms, and **Ł**, **G**, and Π are emerging in this respect as fundamental examples of logics based on continuous t-norms. Esteva and Godo further [5] introduced the logic of *left-continuous* t-norms **MTL** (Monoidal t-norm logic), which copes with the logic of *left-continuous* t-norms, as a weakening of **BL**. This is the most basic t-norm logic known to us. In this approach, (multiplicative) conjunction connectives are interpreted by t-norms (see [11]), which are commutative, associative, monotonic binary functions with identity 1.

Although t-norms play an important role in fuzzy logic (theory), these operators do not admit a compensating behavior. As Detyniecki [3] mentioned, t-norms do not allow low values to be compensated by high values (see [19]). For this reason, Yager and Rybalov [21] introduced *uninorms* as a generalization of t-norms. These operators have identity lying anywhere in [0, 1] rather than at 1 as t-norms. After their introducing uninorms, many interesting properties of uninorms and their applications such as full reinforcement, compensation behavior, bipolar problems, etc., have been studied (see e.g. [1, 7, 13, 17, 19, 20]). Furthermore, several uninorm (based) logics have been recently introduced. For instance, Metcalfe (and Montagna) [14, 15] introduced the uninorm (based) logics **UL**, **IUL** (Involutive uninorm logic), **UML** (Uninorm mingle logic), and **IUML** (Involutive uninorm mingle logic) as substructural fuzzy logics based on *uninorms*. In particular, **UL** is the most basic uninorm logic, which is the logic of conjunctive *left-continuous* uninorms.

Notice that all of the systems above are complete (so called standard complete) w.r.t. algebras with lattice reduct [0, 1]. One method introduced in [6, 12] for **MTL** and its axiomatic extensions (calling it *Jenei and Montagna's method, briefly JM method*), consists of showing that countable linearly ordered algebras of a given variety can be embedded into linearly and *densely* ordered members of the same variety, which can in turn be embedded into algebras with lattice reduct [0, 1]. (Notice that the present author showed that standard completeness for some axiomatic extensions of **UL** using JM method in [22].) But this method seems to fail with associativity for **UL**, and so appears not to work in general for weakening-free fuzzy logics such as **UL** based on uninorms. Because of this negative fact Metcalfe and Montagna [15] instead introduced a new approach for proving standard completeness of uninorm logics (calling it *Metcalfe and Montagna's method, briefly MM method*), consisting of the following two steps: 1. after extending logics with density rule, showing that such systems are complete w.r.t. linearly and densely ordered algebras, and for particular extensions are complete w.r.t. those algebras with lattice reduct [0, 1]; 2. giving a syntactic elimination of density rule (as a rule of the corresponding hypersequent calculus), i.e., showing that if ϕ is derivable in a uninorm logic **L** extended with density rule, then it is also derivable in **L**.

The starting point for the current work is the observation that MM method is unnecessarily complicate. Namely, MM method may be simplified. To verify this, we shall provide a way to simplify MM method by eliminating the step extending logics with density rule. More exactly, we establish a new proof of standard completeness for **UL** by means of a way requiring dense theory in place of density rule. For this we further use nuclear completions method introduced in [8, 9], generalizing Dedekind-McNeille completions.

For convenience, we shall adopt the notation and terminology similar to those in [2, 5, 6, 11, 15], and assume being familiar with them (together with results found in them).

2. Syntax

We base the uninorm logic **UL** on a countable propositional language with formulas *FOR* built inductively as usual from a set of propositional variables *VAR*, binary connectives $\to$, &, $\wedge$, $\vee$, and constants **T**, **F**, **f**, **t**, with defined connectives:

df1. $\sim\phi := \phi \to \mathbf{f}$, and
df2. $\phi \leftrightarrow \psi := (\phi \to \psi) \wedge (\psi \to \phi)$.

We may define **t** as $\mathbf{f} \to \mathbf{f}$. We moreover define ϕ^{n}_{t} as ϕ_t & $\cdots$ & ϕ_t, n factors, where $\phi_t := \phi \wedge \mathbf{t}$. For the remainder we shall follow the customary notation and terminology. We use the axiom systems to provide a consequence relation.

We start with the following axiomatization of **UL** as a (substructural) fuzzy logic.

Definition 2.1 **UL** consists of the following axiom schemes and rules:

A1. $\phi \to \phi$ (self-implication, SI)
A2. $(\phi \wedge \psi) \to \phi$, $(\phi \wedge \psi) \to \psi$ ($\wedge$-elimination, $\wedge$-E)
A3. $((\phi\to\psi)\wedge(\phi\to\chi)) \to (\phi\to(\psi\wedge\chi))$ ($\wedge$-introduction, $\wedge$-I)
A4. $\phi \to (\phi \vee \psi)$, $\psi \to (\phi \vee \psi)$ ($\vee$-introduction, $\vee$-I)
A5. $((\phi\to\chi)\wedge(\psi\to\chi)) \to ((\phi\vee\psi)\to\chi)$ ($\vee$-elimination, $\vee$-E)
A6. $\phi \to \mathbf{T}$ (verum ex quolibet, VE)
A7. $\mathbf{F} \to \phi$ (ex falso quadlibet, EF)
A8. $(\phi \,\&\, \psi) \to (\psi \,\&\, \phi)$ (&-commutativity, &-C)
A9. $(\phi \,\&\, \mathbf{t}) \leftrightarrow \phi$ (push and pop, PP)
A10. $(\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))$ (suffixing, SF)
A11. $(\phi \to (\psi \to \chi)) \leftrightarrow ((\phi \,\&\, \psi) \to \chi)$ (residuation, RE)

A12. for each n, $(\phi \to \psi)^n_t \vee (\psi \to \phi)^n_t$ (^n_t-prelinearity, PL^n_t).

$\phi \to \psi, \phi \vdash \psi$ (modus ponens, mp)

$\phi, \psi \vdash \phi \wedge \psi$ (adjunction, adj)

Proposition 2.2 UL proves:

(1) $(\phi \& (\psi \& \chi)) \leftrightarrow ((\phi \& \psi) \& \chi)$ (&-associativity, AS).

In **UL**, **f** can be defined as $\sim$**t** and vice versa. A *theory* over **UL** is a set T of formulas. A *proof* in a sequence of formulas whose each member is either an axiom of **UL** or a member of T or follows from some preceding members of the sequence using the rules (mp) and (adj). $T \vdash \phi$, more exactly $T \vdash_{UL} \phi$, means that ϕ is *provable* in T w.r.t. **UL**, i.e., there is a **UL**-proof of ϕ in T. The local **t**-deduction theorem (LDT_t) for **UL** is as follows:

Proposition 2.3 Let T be a theory, and ϕ, ψ formulas. $T \cup \{\phi\} \vdash_{UL} \psi$ iff there is n such that $T \vdash_{UL} \phi^n_t \to \psi$.

Proof: See [16]. □

A theory T is *inconsistent* if $T \vdash$ **F**; otherwise it is *consistent*.

For convenience, "$\sim$", "$\wedge$", "$\vee$", and "$\to$" are used ambiguously as propositional connectives and as algebraic operators, but context should make their meaning clear.

3. Semantics

Suitable algebraic structures for **UL** are obtained as a subvariety of the variety of commutative residuated lattices in the sense of e.g. [8].

Definition 3.1 A *pointed bounded commutative residuated lattice* is a structure **A** = (A, $\top$, $\bot$, $\top_t$, $\bot_f$, $\wedge$, $\vee$, *, $\rightarrow$) such that:

(I) (A, $\top$, $\bot$, $\wedge$, $\vee$) is a bounded lattice with top element $\top$ and bottom element $\bot$.

(II) (A, *, $\top_t$) satisfies for some $\top_t$ and for all x, y, z $\in$ A,

(a) $x * y = y * x$ (commutativity)

(b) $\top_t * x = x$ (identity)

(c) $x * (y * z) = (x * y) * z$ (associativity).

(III) $y \leq x \rightarrow z$ iff $x * y \leq z$, for all x, y, z $\in$ A (residuation).

(A, *, $\top_t$) satisfying (II-b, c) is a *monoid*. Thus (A, *, $\top_t$) satisfying (II-a, b, c) is a commutative monoid. To define the above lattice we may take in place of (III) a family of equations as in [11]. Using $\rightarrow$ and $\bot_f$ we can define $\top_t$ as $\bot_f \rightarrow \bot_f$, and $\sim$ as in (df1). Then, UL-algebra whose class characterizes **UL** is defined as follows.

Definition 3.2 (UL-algebra) A *UL-algebra* is a pointed bounded commutative residuated lattice satisfying the condition: for all x, y, and for each n ($\geq$ 1),

(pl$_t$) $\top_t \leq (x \rightarrow y)^n_{\top t} \vee (y \rightarrow x)^n_{\top t}$.

UL-algebra is said to be *linearly ordered* if the ordering of its algebra is linear, i.e., $x \leq y$ or $y \leq x$ (equivalently, $x \wedge y = x$ or $x \wedge y = y$) for each pair x, y.

Definition 3.3 (Evaluation) Let $\mathcal{A}$ be an algebra. An *$\mathcal{A}$-evaluation* is a function v : FOR $\rightarrow \mathcal{A}$ satisfying:

$v(\phi \rightarrow \psi) = v(\phi) \rightarrow v(\psi)$,
$v(\phi \wedge \psi) = v(\phi) \wedge v(\psi)$,
$v(\phi \vee \psi) = v(\phi) \vee v(\psi)$,
$v(\phi \,\&\, \psi) = v(\phi) * v(\psi)$,
$v(\mathbf{F}) = \bot$,
$v(\mathbf{f}) = \bot_{\mathbf{f}}$,

(and hence $v(\sim\phi) = \sim v(\phi)$, $v(\mathbf{T}) = \top$, and $v(\mathbf{t}) = \top_{\mathbf{t}}$).

Definition 3.4 Let $\mathcal{A}$ be a UL-algebra, T a theory, ϕ a formula, and K a class of UL-algebras.

(i) (Tautology) ϕ is a *$\top_t$-tautology* in $\mathcal{A}$, briefly an *$\mathcal{A}$-tautology* (or *$\mathcal{A}$-valid*), if $v(\phi) \geq \top_{\mathbf{t}}$ for each $\mathcal{A}$-evaluation v.

(ii) (Model) An $\mathcal{A}$-evaluation v is an *$\mathcal{A}$-model* of T if $v(\phi) \geq \top_{\mathbf{t}}$ for each $\phi \in$ T. By *Mod(T, $\mathcal{A}$)*, we denote the class of $\mathcal{A}$-models of T.

(iii) (Semantic consequence) ϕ is a *semantic consequence* of T w.r.t. K, denoting by $T \vDash_{\mathsf{K}} \phi$, if $Mod(T, \mathcal{A}) = Mod(T \cup \{\phi\}, \mathcal{A})$ for each $\mathcal{A} \in \mathsf{K}$.

Definition 3.5 (**UL**-algebra) Let $\mathcal{A}$, T, and φ be as in Definition 3.4. $\mathcal{A}$ is a ***UL**-algebra* iff whenever φ is UL-provable in T (i.e. T $\vdash_{\mathbf{UL}}$ φ), it is a semantic consequence of T w.r.t. the set $\{\mathcal{A}\}$ (i.e. T $_{\}}$ φ), $\mathcal{A}$ a UL-algebra. By $MOD^{(l)}(\mathbf{UL})$, we denote the class of (linearly ordered) **UL**-algebras. Finally, we write T $\vDash^{(l)}_{\mathbf{UL}}$ φ in place of T $\vDash_{MOD^{(l)}(\mathbf{UL})}$ φ.

Note that since each condition for the UL-algebra has a form of equation or can be defined in equation (exercise), it can be ensured that the class of all UL-algebras is a variety.

Let **A** be a UL-algebra. We first show that classes of provably equivalent formulas form a UL-algebra. Let T be a fixed theory over **UL**. For each formula φ, let $[\phi]_T$ be the set of all formulas ψ such that T $\vdash_{\mathbf{UL}}$ φ ↔ ψ (formulas T-provably equivalent to φ). A_T is the set of all the classes $[\phi]_T$. We define that $[\phi]_T \to [\psi]_T = [\phi \to \psi]_T$, $[\phi]_T * [\psi]_T = [\phi \,\&\, \psi]_T$, $[\phi]_T \wedge [\psi]_T = [\phi \wedge \psi]_T$, $[\phi]_T \vee [\psi]_T = [\phi \vee \psi]_T$, $\bot = [\mathbf{F}]_T$, $\top = [\mathbf{T}]_T$, $\top_{\mathbf{t}} = [\mathbf{t}]_T$, and $\bot_{\mathbf{f}} = [\mathbf{f}]_T$. By $\boldsymbol{A_T}$, we denote this algebra.

Proposition 3.6 For T a theory over **UL**, $\mathbf{A}_T$ is a **UL**-algebra.

Proof: Note that A1 to A7 ensure that ∧ and ∨ satisfy (I) in Definition 3.1; that AS, A8, A9 ensure that & satisfies (II); that A11 and A12 ensure that (III) and ($pl^n_{\mathbf{t}}$) hold. It is obvious that $[\phi]_T \leq [\psi]_T$ iff T $\vdash_{\mathbf{UL}}$ φ ↔ (φ ∧ ψ) iff T $\vdash_{\mathbf{UL}}$ φ → ψ. Finally recall that $\mathbf{A}_T$ is a **UL**-algebra iff T $\vdash_{\mathbf{UL}}$ ψ implies T $\vDash_{\mathbf{UL}}$ ψ, and observe that for φ in T, since T $\vdash_{\mathbf{UL}}$ **t** → φ, it follows that $[\mathbf{t}]_T \leq [\phi]_T$. Thus it is a **UL**-algebra. □

We next note that the nomenclature of the prelinearity condition is explained by the subdirect representation theorem below.

Proposition 3.7 ([18]) Each UL-algebra is a subdirect product of linearly ordered UL-algebras.

Theorem 3.8 (Strong completeness) Let T be a theory, and ϕ a formula. $T \vdash_{\mathbf{UL}} \phi$ iff $T \models_{\mathbf{UL}} \phi$ iff $T \models^1_{\mathbf{UL}} \phi$.

Proof: (i) $T \vdash_{\mathbf{UL}} \phi$ iff $T \models_{\mathbf{UL}} \phi$. Left to right follows from definition. Right to left is as follows: from Proposition 3.6, we obtain $\mathbf{A}_T \in$ MOD(L), and for $\mathbf{A}_T$-evaluation v defined as $v(\psi) = [\psi]_T$, it holds that $v \in$ Mod(T, $\mathbf{A}_T$). Thus, since from $T \models_{\mathbf{UL}} \phi$ we obtain that $[\phi]_T = v(\phi) \geq \top_{\mathbf{t}}$, $T \vdash_{\mathbf{UL}} \mathbf{t} \rightarrow \phi$. Then, since $T \vdash_{\mathbf{UL}} \mathbf{t}$, by (mp) $T \vdash_{\mathbf{UL}} \phi$, as required.

(ii) $T \models_{\mathbf{UL}} \phi$ iff $T \models^1_{\mathbf{UL}} \phi$. It follows from Proposition 3.7. □

4. Uninorms and their residua

In this section, using *1, 0, and some 1_t, and 0_f* in the real unit interval [0, 1], we shall express $\top$, $\bot$, $\top_{\mathbf{t}}$, and $\bot_{\mathbf{f}}$, respectively. We also define standard UL-algebras and uninorms on [0, 1].

Definition 4.1 A UL-algebra is *standard* iff its lattice reduct is [0, 1].

In standard UL-algebras the monoid operator * is a uninorm.

Definition 4.2 A *uninorm* is a function ○ : $[0, 1]^2 \rightarrow [0, 1]$ such that for some $1_t \in [0, 1]$ and for all x, y, z $\in$ [0, 1]:

(a) x ○ y = y ○ x (commutativity),
(b) x ○ (y ○ z) = (x ○ y) ○ z (associativity),
(c) x $\leq$ y implies x ○ z $\leq$ y ○ z (monotonicity), and
(d) 1_t ○ x = x (identity).

The function ○ satisfying (1-identity) $1_t = 1$ is a *t-norm*. ○ is *residuated* iff there is $\rightarrow : [0, 1]^2 \rightarrow [0, 1]$ satisfying (residuation) on [0, 1]. A uninorm is called *conjunctive* if 0 ○ 1 = 0, and *disjunctive* if 0 ○ 1 = 1.

The left-continuity property of conjunctive uninorms is important in the sense that it gives a residuated implication and so plays an important role in standard completeness proof of **UL** as in t-norm based logics such as **MTL**. Given a uninorm ○, *residuated implication* → determined by ○ is defined as x → y := sup{z: x ○ z $\leq$ y} for all x, y $\in$ [0, 1]. Then, we can show that for any uninorm ○, ○ and its residuated implication → form a residuated pair iff ○ is conjunctive and left-continuous in both arguments (see Proposition 5.4.2 [10]).

5. Standard completeness

We here provide *standard* completeness results for **UL** using nuclear completions in [8, 9]. We shall call these completions method *nuclear completions method.*

A linear theory T is said to be *dense* if for each pair ϕ, ψ of formulas, $T \nvdash \phi \to \psi$ implies that there is a propositional variable p not occurring in T, ϕ, or ψ such that $T \nvdash \phi \to p$ and $T \nvdash p \to \psi$.

Proposition 5.1 Let T be a theory over **UL** and ϕ a formula. $T \vdash_{\mathbf{UL}} \phi$ iff for every linearly densely ordered **UL**-algebra and evaluation v, if $v(\psi) \geq \top_{\mathbf{t}}$ for each $\psi \in T$, then $v(\phi) \geq \top_{\mathbf{t}}$.

Proof: Left to right is by induction on the height of a proof for $T \vdash_{\mathbf{UL}} \phi$. As an example we prove the rule mp. Suppose toward contradiction that there is a linearly and densely ordered **L**-algebra and evaluation v such that $v(\alpha) \geq \top_{\mathbf{t}}$ for each $\alpha \in T$ and $\top_{\mathbf{t}} \leq v(\phi \to \psi)$, $v(\phi)$ but $v(\psi) < \top_{\mathbf{t}}$. Since $v(\phi \to \psi) = v(\phi) \to v(\psi)$, $\top_{\mathbf{t}} \leq v(\phi \to \psi) = v(\phi) \to v(\psi)$ and so $v(\phi) \leq v(\psi)$. This implies that $\top_{\mathbf{t}} \leq v(\psi)$, a contradiction.

We prove right to left contrapositively. We extend the language (if necessary) with countably many new variables not occurring in T or ϕ. We then fix an enumeration (ϕ_n, ψ_n), $n \in \omega$, of all pairs of formulas of the extended language. For a theory T over **UL** such that $T \nvdash_{\mathbf{UL}} \phi$, we define a sequence of sets T_n by induction as follows:

$T_1 = \{\phi' : T \vdash_{\mathbf{UL}} \phi'\}$.
$T_{i+1} = T \cup \{\phi_i \to \psi_i\}$ if $T, \phi_i \to \psi_i \nvdash_{\mathbf{UL}} \phi$,
$T \cup \{\psi_i \to \phi_i\}$ otherwise,

where $T_{i+1} \vdash_{\mathbf{UL}} \phi_i \to \psi_i$ iff for every q_i not in $T_{i+1} \cup \{\phi_i, \psi_i\}$, $T_{i+1} \vdash_{\mathbf{UL}} \phi_i \to q_i$ or $T_{i+1} \vdash_{\mathbf{UL}} q_i \to \psi_i$.

Let T' be the union of all these T_n's. By Proposition 3.6, $\mathbf{A}_{T'}$ is a **UL**-algebra. Moreover, $\mathbf{A}_{T'}$ is linearly and densely ordered. For this we show that T' is linearly and densely ordered. For linearity, it suffices to note that having $T_n \nvdash_{\mathbf{UL}} \phi$ observe that $T, \phi_i \to \psi_i \nvdash_{\mathbf{UL}} \phi$ or $T, \psi_i \to \phi_i \nvdash_{\mathbf{UL}} \phi$. Otherwise, $T, \phi_i \to \psi_i \vdash_{\mathbf{UL}} \phi$ and $T, \psi_i \to \phi_i \vdash_{\mathbf{UL}} \phi$. Then by $\mathrm{LDT}_{\mathbf{t}}$, there are m, n such that $T \vdash_{\mathbf{UL}} (\phi_i \to \psi_i)^m_{\mathbf{t}} \to \phi$ and $T \vdash_{\mathbf{UL}} (\psi_i \to \phi_i)^n_{\mathbf{t}} \to \phi$. Since $(\phi_{\mathbf{t}} \,\&\, \phi_{\mathbf{t}}) \to \phi_{\mathbf{t}}$, without loss of generality we may assume that $m \leq n$ and so $T \vdash_{\mathbf{UL}} (\phi_i \to \psi_i)^n_{\mathbf{t}} \to \phi$ and $T \vdash_{\mathbf{UL}} (\psi_i \to \phi_i)^n_{\mathbf{t}} \to \phi$. Then, by adj, $T \vdash_{\mathbf{UL}} ((\phi_i \to \psi_i)^n_{\mathbf{t}} \to \phi) \wedge ((\psi_i \to \phi_i)^n_{\mathbf{t}} \to \phi)$, and so by A5 and mp, $T \vdash_{\mathbf{UL}} ((\phi_i \to \psi_i)^n_{\mathbf{t}} \vee (\psi_i \to \phi_i)^n_{\mathbf{t}}) \to \phi$. But then by A12, $T \vdash_{\mathbf{UL}} \phi$, a contradiction. For density, we just note that it follows from the definition that if $T' \nvdash_{\mathbf{UL}} \phi_n \to \psi_n$, then $T' \nvdash_{\mathbf{UL}} \phi_n \to q_n$ and $T' \nvdash_{\mathbf{UL}} q_n \to \psi_n$; and if $T' \nvdash_{\mathbf{UL}} \psi_n \to \phi_n$, then $T' \nvdash_{\mathbf{UL}} \psi_n \to q_n$ and $T' \nvdash_{\mathbf{UL}} q_n \to \phi_n$.

Hence, defining an evaluation v such that $v(p) = [p]_{T'}$ for all propositional variables p, we obtain that $v(\psi) = [\psi]_{T'} \geq \top_{\mathbf{t}}$ for each $\psi \in T'$, but $v(\phi) = [\phi]_{T'} < \top_t$, as desired. □

A *partially-ordered monoid* (*po-monoid* for brevity) is a structure **A** = (A, ≤, *) such that * is a binary operation on A, ≤ is a partial order on A, and * is order preserving, i.e., monotone. A (commutative) residuated lattice is a po-monoid. A *nucleus* on a po-monoid **A** is a map g : A → A such that g is a closure operator on (A, ≤) and for all x, y ∈ A,

(nuc) $g(x) * g(y) \leq g(x * y)$.

Using nuclear completions we show that **UL** is standard complete.

Theorem 5.2 Every countable linearly and densely ordered **UL**-algebra can be embedded into a standard **UL**-algebra.

Proof: Its proof is analogous to that of Theorem 28 in [15]. We first recall that any (bounded and pointed) residuated lattice **A** can be embedded into a complete residuated lattice $\mathbf{A}^+$ by means of the nuclear completion (see [8]). The lattice $\mathbf{A}^+$ is defined as follows:

1. For every X ⊆ **A**, let C(X) denote the intersection of all sets Z such that: (1) X ⊆ Z, (2) Z is closed downward, and (3) for all Y ⊆ Z, if sup(Y) exists in **A**, then sup(Y) ∈ Z. Then it follows that C is a closure operator. The domain of $\mathbf{A}^+$ is {X: X

$\subseteq$ **A** such that C(X) = X}.

2. The operations of $\mathbf{A}^+$ are: X ○ Y = C(X * Y), where, letting * be the monoid operator of **A**, X * Y = {x * y: x ∈ X and y ∈ Y}; X ∧ Y = X ∩ Y; X ∨ Y = C(X ∪ Y); and X → Y = {z ∈ **A**: ∀x ∈ X, z * x ∈ Y}. Then it follows from the definition that C is a nucleus on ($\mathbf{A}^+$, ⊆) because C(X) ○ C(Y) = X ○ Y = C(X ○ Y) for X, Y ∈ $\mathbf{A}^+$.

3. The constants in $\mathbf{A}^+$ are: $\top^+$ = **A**, $\bot^+$ = {⊥}, $\top_{\mathbf{t}}^+$ = {z ∈ **A**: z ≤ $\top_{\mathbf{t}}$}, and $\bot_{\mathbf{f}}^+$ = {z ∈ **A**: z ≤ $\bot_{\mathbf{f}}$}.

First note that $\mathbf{A}^+$ is the nucleus retraction of **A**. The embedding h of **A** into $\mathbf{A}^+$ is defined by h(x) = {z ∈ **A**: z ≤ x}. Notice that for X ∈ $\mathbf{A}^+$, we have X = sup{h(x): x ∈ X}, i.e., every element of $\mathbf{A}^+$ is the supremum of a set of elements of **A**. Furthermore, the suprema and infima existing in **A** are preserved by h, and for X, Y ∈ $\mathbf{A}^+$,

(1) X ○ Y = sup{h(x) ○ h(y): x ∈ X, y ∈ Y}.

Since C-closed sets are closed downwards and so C is a downward nucleus, if **A** is linearly ordered, so is $\mathbf{A}^+$ by inclusion. Hence, if **A** is a linearly ordered **UL**-algebra, so is $\mathbf{A}^+$. Note that if **A** is densely ordered, the image of **A** under h is dense in $\mathbf{A}^+$, i.e., for every X ⊂ Y ∈ $\mathbf{A}^+$, there is z ∈ **A** such that X ⊂ h(z) ⊂ Y. Hence, if **A** is a countable linearly and densely ordered **UL**-algebra, it is order isomorphic to **Q** ∩ [0, 1], and its nuclear completion, be completely and densely ordered, is isomorphic to [0, 1]. Since by (1), the monoid operation $\circ$ on $\mathbf{A}^+$ is left-continuous, it follows that $\mathbf{A}^+$ is a standard **UL**-algebra. □

Theorem 5.3 (Strong standard completeness) $T \vdash_{\mathbf{UL}} \phi$ iff for every standard **UL**-algebra and evaluation v, if $v(\psi) \geq \top_{\mathbf{t}}$ for each $\psi \in T$, then $v(\phi) \geq \top_{\mathbf{t}}$.

Proof: It follows from Proposition 5.1 and Theorem 5.2. □

Remark 5.4 Recall that any (bounded and pointed) residuated lattice **A** can be embedded into a complete residuated lattice $\mathbf{A}^+$ by means of the Dedekind-McNeille completion (see [8]). This implies that we can prove standard completeness of **UL** using Dedekind-McNeille completion in place of nuclear completion. We here just note that Theorem 5.2 can be proved using Dedekind-McNeille completion (see Theorem 28 in [15]), and this gives a standard completeness of **UL** using Dedekind-McNeille completion.

6. Concluding remark

We here investigated (not merely algebraic completeness but also) standard completeness for **UL**. This work can be generalized to the systems, which are the axiomatic extensions of **UL** introduced in [15]. We shall investigate this in some subsequent paper.

To some readers it will be interesting to say that **IUML**, an extension of **UL**, is **R**-mingle (**RM**) plus (FP) **t** ↔ **f** and so **IUML** can be regarded not merely as fuzzy logic but also as relevance logic. Dunn (see e.g. [4]) provided a Kripke-style semantics for **RM** and Yang (see [23]) has recently studied Kripke-style semantics for some neighbors of **R**. Kripke-style semantics seems to be provided for **UL** and its axiomatic extensions, in particular, **IUML**. We shall consider this in another subsequent paper.

References

[1] Bisrarelli, S., Pini, M. S., Montanari, U., Rossi, F., and Venable, K. B., "Bipolar preference problems", *Frontiers in Artificial Intelligence and Applications*, Vol. 141, Amsterdam, IOS Press, 2006, pp. 705-706.

[2] Cintula, P., "Weakly Implicative (Fuzzy) Logics I: Basic properties", *Archive for Mathematical Logic*, (2006), pp. 673-704.

[3] Detyniecki, M., **Fundamentals on Aggregation Operators**, Asturias, 2001.

[4] Dunn, J. M., "Partiality and its Dual", *Studia Logica*, 66 (2000), pp. 5-40.

[5] Esteva, F., and Godo, L., "Monoidal t-norm based logic: towards a logic for left-continous t-norms", *Fuzzy Sets and Systems*, 124 (2001), pp. 271-288.

[6] Esteva, F., Gispert, L., Godo, L., and Montagna, F., "On the standard and rational completeness of some axiomatic extensions of the monoidal t-norm logic", *Studia Logica*, 71 (2002), pp. 393-420.

[7] Fordor, J. C., Yager, R. R., and Rybalov, A., "Structure of uninorms", *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 6 (1997), pp. 411-427.

[8] Galatos, N., Jipsen, P., Kowalski, T., and Ono, H., **Residuated lattices: an algebraic glimpse at substructural logics**, Amsterdam, Elsevier, 2007.

[9] Galatos, N. and Ono, H., "Cut elimination and strong

separation for substructural logics", *Annals of Pure and Applied Logic*, In press.

[10] Gottwald, S., **A Treatise on Many-valued Logics**, Baldock, Research studies press LTD., 2001.

[11] Hájek, P., **Metamathematics of Fuzzy Logic**, Amsterdam, Kluwer, 1998.

[12] Jenei, S and Montagna, F., "A Proof of Standard completeness for Esteva and Godo's Logic MTL", *Studia Logica*, 70 (2002), pp. 183-192.

[13] Li, Y., and Shi, Z., "Remarks on uninorm aggregation operators", *Fuzzy Sets and Systems*, 114 (2000), pp. 377-380.

[14] Metcalfe, G., "Uninorm Based Logics", *Proceedings of EUROFUSE*, 2004, pp. 85-99, Exit Press, 2004.

[15] Metcalfe, G and Montagna, F., "Substructural Fuzzy Logics", *Journal of Symbolic Logic*, 72 (2007), pp. 834-864.

[16] Novak, V., "On the syntactico-semantical completeness of first-order fuzzy logic I, II", *Kybernetika*, 26 (1990), pp. 47-66.

[17] Rudas, I., and Fordor, J. C., "Information aggregation in intelligent systems using generalized operators", *International Journal of Computers, Communications and Control*, 1 (2006), pp. 47-57.

[18] Tsinakis, C., and Blount, K., "The structure of residuated lattices", *International Journal of Algebra and Computation*, 13 (2003), pp. 437-461.

[19] Yager, R. R., "Full reinforcement operators in aggregation techniques", *IEEE Transactions on Systems, Man and*

Cybernetics, 28 (1998), pp. 757-769.

[20] Yager, R. R., "Uninorms in fuzzy systems modeling", *Fuzzy Sets and Systems*, 122 (2001), pp. 167-175.

[21] Yager, R. R., and Rybalov, A., "Uninorm aggregation operators", *Fuzzy Sets and Systems*, 80 (1996), pp. 111-120.

[22] Yang, E., "On the standard completeness of an axiomatic extension of the uninorm logic", *Korean Journal of Logic*, 12 (2009), pp. 115-139.

[23] Yang, E., "(Star-based) four-valued Kripke-style Semantics for some neighbors of **E**, **R**, **T**", *Logique et Analyse*, 207 (2009), pp. 255-280.

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