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Some axiomatic extensions of the involutive mianorm Logic **IMIAL***

Eunsuk Yang

【Abstract】In this paper, we deal with standard completeness of some axiomatic extensions of the involutive mianorm logic **IMIAL**. More precisely, first, seven involutive mianorm-based logics are introduced. Their algebraic structures are then defined, and their corresponding algebraic completeness is established. Next, standard completeness is established for four of them using construction in the style of Jenei-Montagna.

【Abstract】fuzzy logic, involution, mianorm, algebraic completeness, standard completeness, **IMAL**

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1. Introduction

The present author recently introduced *mianorms* (binary monotonic identity aggregation operations on the real unit interval [0, 1]) and logics based on mianorms in Yang (2016). In particular, he (2017a) provided standard completeness results for involutive such logics, which was a problem left open in Horčík (2011) and Cintula et al. (2013), using the Jenei-Montagna-style construction introduced in Esteva et al. (2002) and Jenei & Montagna (2002). After providing such completeness, he stated as follows in Remark 2:

> The proof of standard completeness in Theorem 4 does not work for **IMIAL**c, **IMIAL**$c^r{}_n$, ,**IMIAL**$c^l{}_n$, **IMIAL**cp and **IMIAL**$c^r{}_n c^l{}_n$ because the definition of ⊙ does not satisfy contraction. ⋯ However, if each of **IMIAL**$c^r{}_n$, ,**IMIAL**$c^l{}_n$, **IMIAL**cp and **IMIAL**$c^r{}_n c^l{}_n$ has, in addition, the rule f (but the carrier set A has at least three elements), such case does not happen; therefore, we can provide standard completeness for **IMIAL**cf, **IMIAL**$c^r{}_n$f, **IMIAL**$c^l{}_n$f, **IMIAL**cpf and **IMIAL**$c^r{}_n c^l{}_n$f(Yang (2017a), p. 11).

Let ϕ^n and $^n\phi$ stand for $((\cdots(\phi \,\&\, \phi) \,\&\, \cdots) \,\&\, \phi) \,\&\, \phi$, n ϕ's and $\phi \,\&\, (\phi \,\&\, \cdots \,\&\, (\phi \,\&\, \phi)\cdots))$, n ϕ's, respectively. Consider the structural axioms introduced in Proposition 2 below. The systems **IMIAL**cw, **IMIAL**cf, **IMIAL**$c^r{}_n$w, **IMIAL**$c^l{}_n$w, **IMIAL**$c^r{}_n$f, **IMIAL**$c^l{}_n$f, and **IMIAL**cpf are the **IMIAL** with the corresponding structural axioms. For instance, the system **IMIAL**cw is the **IMIAL** with the axioms c and w. As the statements in Remark 2 of Yang (2017a) show, although he insists that the proof in Theorem 4 (the standard completeness using the construction in the style of Jenei-Montagna) of Yang (2017a) is applicable to the systems, he does not provide concrete proofs for the systems.

In this paper, we show that some caution needs in proving standard completeness because if we drop the condition that the carrier set A has at least three elements, the claim in Remark 2 of Yang (2017a) is not applicable to some particular systems. For this, we verify that the proof of Theorem 4 in Yang (2017a) is applicable to the systems **IMIAL**cf, **IMIAL**c^r_nf, **IMIAL**c^l_nf, and **IMIAL**cpf, but not to **IMIAL**cw, **IMIAL**c^r_nw, and **IMIAL**c^l_nw if we eliminate the above condition. Our results show that, while the systems **IMIAL**c, **IMIAL**c^r_n, **IMIAL**c^l_n, and **IMIAL**cp are not standard complete, such systems with the axiom f are standard complete.

The paper is organized as follows. In Section 2, we present the axiomatizations of the systems **IMIAL**cw, **IMIAL**cf, **IMIAL**c^r_nw, **IMIAL**c^l_nw, **IMIAL**c^r_nf, **IMIAL**c^l_nf, and **IMIAL**cpf, define their corresponding algebraic structures, by subvarieties of the variety of residuated lattices, and show that they are complete with respect to (w.r.t.) linearly ordered corresponding algebras. In Section 3, we establish standard completeness for the systems except for **IMIAL**cw, **IMIAL**c^r_nw, and **IMIAL**c^l_nw using the method introduced in Yang (2017a).

For convenience, we shall adopt notations and terminology similar to those in Cintula (2006), Esteva et al. (2002), Hájek (1998), Metcalfe & Montagna (2007), Yang (2009; 2013, 2014, 2015, 2016, 2017a), and assume familiarity with them (together with the results found therein).[1)]

2. Syntax

We base some axiomatic extensions of the involutive mianorm logic **IMIAL** on a countable propositional language with formulas *Fm* built inductively as usual from a set of propositional variables *VAR*, binary connectives $\rightarrow$, $\Rightarrow$, &, $\wedge$, $\vee$, and constants **T**, **F**, **f**, **t** with defined connectives:

df1. $\neg\phi := \phi \rightarrow \mathbf{f}$,
df2. $\sim\phi := \phi \Rightarrow \mathbf{f}$, and
df3. $\phi \leftrightarrow \psi := (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$.

We may define **t** as $\mathbf{f} \rightarrow \mathbf{f}$. We moreover define $\phi^n_{\mathbf{t}}$ as $\phi_{\mathbf{t}}$ & $\cdots$ & $\phi_{\mathbf{t}}$, n factors, where $\phi_{\mathbf{t}} := \phi \wedge \mathbf{t}$. For the rest of this paper, we use the customary notations and terminology, and the axiom systems to provide a consequence relation.

We start with the following axiomatization of **IMIAL**, the most basic fuzzy logic introduced here.

Definition 2.1 (Yang (2017a)) **IMIAL** consists of the following axiom schemes and rules:

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1) For some basic concepts and ideas of substrutural logics and fuzzy logics, see Galatos et al. (2007), Cintula & Noguera (2011), and Cintula et al. (2013; 2015).

A1. $(\phi \wedge \psi) \to \phi$, $(\phi \wedge \psi) \to \psi$ ($\wedge$-elimination, $\wedge$-E)

A2. $((\phi\to\psi)\wedge (\phi\to\chi)) \to (\phi\to(\psi\wedge\chi))$ ($\wedge$-introduction, $\wedge$-I)

A3. $\phi \to (\phi \vee \psi)$, $\psi \to (\phi \vee \psi)$ ($\vee$-introduction, $\vee$-I)

A4. $((\phi\to\chi)\wedge (\psi\to\chi)) \to ((\phi\vee\psi)\to\chi)$ ($\vee$-elimination, $\vee$-E)

A5. $\mathbf{F} \to \phi$ (ex falso quadlibet, EF)

A6. $(\mathbf{t} \to \phi) \leftrightarrow \phi$ (push and pop, PP)

A7. $\phi \to (\psi \to (\psi \& \phi))$ (&-adjunction$_{\to}$, &-Adj$_{\to}$)

A8. $\phi \to (\psi \Rightarrow (\phi \& \psi))$ (&-Adj$_{\Rightarrow}$)

A9. $(\phi_{\mathbf{t}} \& \psi_{\mathbf{t}}) \to (\phi \wedge \psi)$ ($\&\wedge$)

A10. $(\psi \& (\phi \& (\phi \to (\psi \to \chi)))) \to \chi$ (residuation, Res')

A11. $((\phi \& (\phi \Rightarrow (\psi \to \chi))) \& \psi) \to \chi$ (Res'$_{\Rightarrow}$)

A12. $((\phi \to (\phi \& (\phi \to \psi))) \& (\psi \to \chi)) \to (\phi \to \chi)$ (T')

A13. $((\phi \Rightarrow ((\phi\Rightarrow\psi) \& \phi))) \& (\psi \to \chi)) \to (\phi \Rightarrow \chi)$ (T''$_{\Rightarrow}$)

A14. $(\phi \to \psi)_{\mathbf{t}} \vee ((\delta \& \varepsilon) \to (\delta \& (\varepsilon \& (\psi\to\phi)_{\mathbf{t}})))$ (PL$\alpha_{\delta,\varepsilon}$)

A15. $(\phi \to \psi)_{\mathbf{t}} \vee ((\delta \& \varepsilon) \to ((\delta \& (\psi\to\phi)_{\mathbf{t}}) \& \varepsilon))$ (PL$\alpha'_{\delta,\varepsilon}$)

A16. $(\phi \to \psi)_{\mathbf{t}} \vee (\delta \to (\varepsilon \to ((\varepsilon\&\delta) \& (\psi \to \phi)_{\mathbf{t}})))$ (PL$\beta_{\delta,\varepsilon}$)

A17. $(\phi \to \psi)_{\mathbf{t}} \vee (\delta \to (\varepsilon \Rightarrow ((\varepsilon\&\delta) \& (\psi\to\phi)_{\mathbf{t}})))$ (PL$\beta'_{\delta,\varepsilon}$)

A14. $\sim\neg\phi \to \phi$ (double negation elimination, DNE(1))

A14. $\neg\sim\phi \to \phi$ (double negation elimination, DNE(2))

$\phi \to \psi, \phi \vdash \psi$ (modus ponens, mp)

$\phi \vdash \phi_{\mathbf{t}}$ (adj$_{u}$)

$\phi \vdash (\delta \& \varepsilon) \to (\delta \& (\varepsilon \& \phi))$ (α)

$\phi \vdash (\delta \& \varepsilon) \to ((\delta \& \phi) \& \varepsilon)$ (α')

$\phi \vdash \delta \to (\varepsilon \to ((\varepsilon \& \delta) \& \phi))$ (β)

$\phi \vdash \delta \to (\varepsilon \Rightarrow ((\delta \& \varepsilon) \& \phi))$ (β')

Definition 2.2 (Yang (2017a)) A logic is an axiomatic extension

(extension for short) of an arbitrary logic **L** if and only if (iff) it results from **L** by adding axiom schemes. Consider the following structural axioms:

c	$\phi \to (\phi \,\&\, \phi)$	contraction
p	$(\phi \,\&\, \phi) \to \phi$	expansion
w	$\phi \to (\psi \to \phi)$	weakening
c^r_n	$\phi^n \leftrightarrow \phi^{n-1}$, $2 \leq n$	right n-potency
c^l_n	${}^n\phi \leftrightarrow {}^{n-1}\phi$, $2 \leq n$	left n-potency
f	$\mathbf{t} \leftrightarrow \mathbf{f}$.	fixed point

In particular, the following are weakening-free, non-associative non-commutative fuzzy logics that extend **IMIAL**:

- **IMIAL**cw is **IMIAL** plus c and w.
- **IMIAL**cf is **IMIAL** plus c and f.
- **IMIAL**c^r_nw is **IMIAL** plus c^r_n and w.
- **IMIAL**c^l_nw is **IMIAL** plus c^l_n and w.
- **IMIAL**c^r_nf is **IMIAL** plus c^r_n and f.
- **IMIAL**c^l_nf is **IMIAL** plus c^l_n and f.
- **IMIAL**cpf is **IMIAL** plus c, p, and f.

For easy reference, we let Ls be the set of the weakening-free, non-associative non-commutative fuzzy logics defined in Definition 2.2.

Definition 2.3 Ls = {**IMIAL**cw, **IMIAL**cf, **IMIAL**c^r_nw, **IMIAL**c^l_nw, **IMIAL**c^r_nf, **IMIAL**c^l_nf, **IMIAL**cpf}

A *theory* over L ($\in$ Ls) is a set T of formulas. A *proof* in a theory over L is a sequence of formulas whose each member is either an axiom of L or a member of T or follows from some preceding members of the sequence using a rule of L. T $\vdash \phi$, more exactly T $\vdash_L \phi$, means that ϕ is *provable* in T w.r.t. L, i.e., there is an L-proof of ϕ in T. A theory T is *inconsistent* if T $\vdash$ **F**; otherwise it is *consistent*.

The deduction theorem for L is as follows:

Proposition 2.4 (Cintula et al. (2013; 2015)) Let T be a theory, and ϕ, ψ be formulas. T $\cup \{\phi\} \vdash_L \psi$ iff T $\vdash_L \gamma(\phi) \rightarrow \psi$ for some $\gamma \in \Pi(bDT^*)$.[2)]

For convenience, "$\neg$," "$\sim$," "$\wedge$," "$\vee$," "$\rightarrow$," and "$\Rightarrow$" are used ambiguously as propositional connectives and as algebraic operators, but context should clarify their meanings.

Suitable algebraic structures for L ($\in$ Ls) are obtained as varieties of residuated lattice-ordered unital groupoids (briefly, rlu-groupoids) in the sense of Galatos et al. (2007).

Definition 2.5 (Yang (2017a)) (i) A *pointed bounded rlu-groupoid* is a structure **A** = (A, $\top$, $\bot$, t, f, $\wedge$, $\vee$, *, $\rightarrow$, $\Rightarrow$) such that:

(Ⅰ) (A, $\top$, $\bot$, $\wedge$, $\vee$) is a bounded lattice with top element $\top$ and bottom element $\bot$.

(Ⅱ) (A, *, t) is a groupoid with unit.

(Ⅲ) $y \leq x \rightarrow z$ iff $x * y \leq z$ iff $x \leq y \Rightarrow z$, for all x, y,

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2) For γ and $\Pi(bDT^*)$, see Cintula et al. (2013; 2015) and Yang (2017a).

$z \in A$ (residuation).

(ii) Let $\neg x = x \rightarrow f$ and $\sim x = x \Rightarrow f$ for all $x \in A$. A pointed, bounded rlu-groupoid is *involutive* if it satisfies the following: for all $x \in A$,

- $\sim\neg x \leq x$ ($DNE(1)^{A}$)
- $\neg\sim x \leq x$ ($DNE(2)^{A}$).

(iii) An *IMIAL-algebra* is a pointed, bounded involutive rlu-groupoid satisfying: for all $x, y, z \in A$,

- $t \leq (x \rightarrow y)_t \vee ((z * w) \rightarrow (z * (w*(y\rightarrow x)_t)))$ ($PL\alpha_{\delta,\varepsilon}{}^{A}$)
- $t \leq (x \rightarrow y)_t \vee ((z * w) \rightarrow ((z*(y\rightarrow x)_t) * w))$ ($PL\alpha'_{\delta,\varepsilon}{}^{A}$)
- $t \leq (x \rightarrow y)_t \vee (z \rightarrow (w \rightarrow ((w*z) * (y\rightarrow x)_t)))$ ($PL\beta_{\delta,\varepsilon}{}^{A}$)
- $t \leq (x \rightarrow y)_t \vee (z \rightarrow (w \Rightarrow ((w*z)*(y\rightarrow x)_t)))$ ($PL\beta'_{\delta,\varepsilon}{}^{A}$).

L-algebras the class of which characterizes L are defined as follows.

Definition 2.6 (L-algebras) The algebraic (in)equations corresponding to the structural axioms introduced in Definition 2.2 are defined as follows: for all $x \in A$,

- $x \leq x * x$ (c^{A})
- $x * x \leq x$ (p^{A})
- $x \leq 1$ (w^{A})
- $x^n = x^{n-1}$, $2 \leq n$ ($c^{r}_{n}{}^{A}$)
- ${}^{n}x = {}^{n-1}x$, $2 \leq n$ ($c^{l}_{n}{}^{A}$)
- $1 = 0$ (f^{A}).

An *IMIALcw-algebra* is an IMIAL-algebra satisfying (c^{A}) and (w^{A}); an *IMIALcf-algebra* is an IMIAL-algebra satisfying (c^{A}) and

($f^{\mathbf{A}}$); an *IMIALc$^{r}_{n}$w-algebra* is an IMIAL-algebra satisfying ($c^{r}_{n}{}^{\mathbf{A}}$) and ($w^{\mathbf{A}}$); an *IMIALc$^{l}_{n}$w-algebra* is an IMIAL-algebra satisfying ($c^{l}_{n}{}^{\mathbf{A}}$) and ($w^{\mathbf{A}}$); an *IMIALc$^{r}_{n}$f-algebra* is an IMIAL-algebra satisfying ($c^{r}_{n}{}^{\mathbf{A}}$) and ($f^{\mathbf{A}}$); an *IMIALc$^{l}_{n}$f-algebra* is an IMIAL-algebra satisfying ($c^{l}_{n}{}^{\mathbf{A}}$) and ($f^{\mathbf{A}}$); an *IMIALcpf-algebra* is an IMIAL-algebra satisfying ($c^{\mathbf{A}}$), ($p^{\mathbf{A}}$), and ($f^{\mathbf{A}}$). We call all these algebras *L-algebras*.

An L-algebra is said to be *linearly ordered* if the ordering of its algebra is linear, i.e., $x \leq y$ or $y \leq x$ (equivalently, $x \wedge y = x$ or $x \wedge y = y$) for each pair x, y.

Definition 2.7 (Evaluation) Let $\mathcal{A}$ be an algebra. An *$\mathcal{A}$-evaluation* is a function v : FOR $\rightarrow \mathcal{A}$ satisfying: $v(\phi \rightarrow \psi) = v(\phi) \rightarrow v(\psi)$, $v(\phi \Rightarrow \psi) = v(\phi) \Rightarrow v(\psi)$, $v(\phi \wedge \psi) = v(\phi) \wedge v(\psi)$, $v(\phi \vee \psi) = v(\phi) \vee v(\psi)$, $v(\phi \,\&\, \psi) = v(\phi) * v(\psi)$, $v(\mathbf{F}) = \bot$, $v(\mathbf{f}) = f$, (and hence $v(\neg\phi) = \neg v(\phi)$, $v({\sim}\phi) = {\sim}v(\phi)$, $v(\mathbf{T}) = \top$, and $v(\mathbf{t}) = t$).

Definition 2.8 (Cintula (2006)) Let $\mathcal{A}$ be an L-algebra, T a theory, ϕ a formula, and K a class of L-algebras.

(i) (Tautology) ϕ is a *t-tautology* in $\mathcal{A}$, briefly an *$\mathcal{A}$-tautology* (or *$\mathcal{A}$-valid*), if $v(\phi) \geq t$ for each $\mathcal{A}$-evaluation v.

(ii) (Model) An $\mathcal{A}$-evaluation v is an *$\mathcal{A}$-model* of T if $v(\phi) \geq t$ for each $\phi \in T$. We denote the class of $\mathcal{A}$-models of T, by *Mod(T, $\mathcal{A}$)*.

(iii) (Semantic consequence) ϕ is a *semantic consequence* of T w.r.t. K, denoting by $T \vDash_K \phi$, if $Mod(T, \mathcal{A}) = Mod(T \cup \{\phi\}, \mathcal{A})$ for each $\mathcal{A} \in$ K.

Definition 2.9 (**L**-algebra, Cintula (2006)) Let $\mathcal{A}$, T, and ϕ be as in Definition 3.4. $\mathcal{A}$ is an ***L**-algebra* iff, whenever ϕ is L-provable in T (i.e. $T \vdash_L \phi$, L an L logic), it is a semantic consequence of T w.r.t. the set $\{\mathcal{A}\}$ (i.e. $T \vDash_{\{A\}} \phi$), $\mathcal{A}$ a corresponding L-algebra). By $MOD^{(l)}(L)$, we denote the class of (linearly ordered) **L**-algebras. Finally, we write $T \vDash^{(l)}_L \phi$ in place of $T \vDash_{MOD^{(l)}(L)} \phi$.

Theorem 2.10 (Strong completeness) Let T be a theory, and ϕ be a formula. $T \vdash_L \phi$ iff $T \vDash_L \phi$ iff $T \vDash^l_L \phi$.

Proof: We obtain this theorem as a corollary of Theorem 3.1.8 in Cintula & Noguera (2011). □

3. Standard completeness

In this section, we provide standard completeness results for L ($\in$ Ls $\setminus$ {**IMIAL**cw, **IMIAL**c^r_nw, **IMIAL**c^l_nw}) using the Jenei-Montagna-style construction in Eeteva et al. (2002) and Jenei & Montagna (2002).

We first show that finite or countable, linearly ordered **L**-algebras are embeddable into a standard algebra. (For convenience, we add the ‘less than or equal to’ relation symbol “$\leq$” to such algebras.) First, note the following results.

Theorem 3.1 (i) (Yang (2016)) For every finite or countable linearly ordered **MIAL**-algebra **A** = (A, $\leq_A$, ⊤, ⊥, t, f, ∧, ∨, *, →, ⇒), there is a countable ordered set X, a binary operation ○, and a map h from A into X such that the following conditions hold:

(Ⅰ) X is densely ordered, and has a maximum Max, a minimum Min, and special elements e, ∂.

(Ⅱ) (X, ○, ≤, e) is a linearly ordered, monotonic groupoid with unit.

(Ⅲ) ○ is conjunctive and left-continuous w.r.t. the order topology on (X, ≤).

(Ⅳ) h is an embedding of the structure (A, $\leq_A$, ⊤, ⊥, t, f, ∧, ∨, *) into (X, ≤, Max, Min, e, ∂, min, max, ○), and for all m, n ∈ A, h(m → n) and h(m ⇒ n) are the residated pair of h(m) and h(n) in (X, ≤, Max, Min, e, ∂, max, min, ○).

(ii) (Yang (2017a)) For every finite or countable linearly ordered **IMIAL**-algebra **A** = (A, $\leq_A$, ⊤, ⊥, t, f, ∧, ∨, *, →, ⇒), there is a countable ordered set X, a binary operation ○, and a map h from A into X such that the conditions (I) to (IV) in (i) and the following condition hold:

(Ⅴ) For all x ∈ X, x is involutive, i.e., it satisfies (DNE(1)$^{\mathbf{A}}$) and (DNE(2)$^{\mathbf{A}}$).

Proposition 3.2 For every finite or countable linearly ordered **IMIAL**cf-algebra (**IMIAL**c^r_nf-algebra, **IMIAL**c^l_nf-algebra, **IMIAL**cpf-algebra, respectively) **A** = (A, $\leq_A$, ⊤, ⊥, t, f, ∧, ∨, *, →, ⇒), there is a countable ordered set X, a binary operation

$\circ$, and a map h from A into X such that the conditions (I) to (V) of (ii) in Theorem 5.1 and the corresponding conditions hold.

Proof: For convenience, we assume A as a subset of **Q** $\cap$ [0, 1] with a finite or countable number of elements, where 0 and 1 are least and greatest elements, respectively, and some e and any ∂ are special elements, each of which corresponds to $\top$, $\bot$, t, f, respectively.

We first note that, for **MIAL**, a linearly ordered, monotonic groupoid with unit (X, $\circ$, $\leq$, e) is defined as follows:

$$X = \{(m, x): m \in A \setminus \{0 (= \bot)\} \text{ and } x \in \mathbf{Q} \cap (0, m]\} \cup \{(0, 0)\};$$

for (m, x), (n, y) $\in$ X,

(m, x) $\leq$ (n, y) iff either m $<_A$ n, or m $=_A$ n and x $\leq$ y;

$$(m,x) \circ (n,y) = \begin{cases} \max\{(m,x), (n,y)\} & \text{if } m*n =_A m \vee n,\ m \neq_A n, \text{ and } (m, x) \leq e \text{ or } (n, y) \leq e; \\ \min\{(m,x), (n,y)\} & \text{if } m * n = m \wedge z, \text{ and } (m, x) \leq e \text{ or } (n, y) \leq e; \\ (m * n, m * n) & \text{otherwise.} \end{cases}$$

For convenience, we henceforth drop the index A in $\leq_A$ and $=_A$, if we need not distinguish them. Context should clarify the intention.

We next note that, for **IMIAL**, m^+ denotes the successor of m if it exists, otherwise $m^+ = m$, for each $m \in A$. Note that, since the pair of negations in A, defined as $\neg m := m \rightarrow \partial$ and $\sim m := m \Rightarrow \partial$, is involutive, we have that: $m = (\neg n)^+$ iff $n = (\neg m)^+$ and $m = (\sim n)^+$ iff $n = (\sim m)^+$; moreover, if $m < m^+$, then $(\neg(m^+))^+ = \neg m$ and $(\sim(m^+))^+ = \sim m$. Here, we use Y below in place of the X above. Let $(Y, \leq)$ be the linearly ordered set, defined by

$$Y = \{(m, m): m \in A\} \cup$$
$$\{(m, x): \exists m' \in A \text{ such that } m = m'^+ > m', \text{ and } x \in Q \cap (0, m)\},$$

and $\leq$ being the corresponding lexicographic ordering as above. It is clear that $(Y, \leq)$ is a subset of the ordered set $(X, \leq)$ defined as above with the same bounds and special elements e (= (t, t)) and ∂ (= (f, f)). Notice that Y is closed under $\circ$ and that $\leq$ is a linear order with maximum (1, 1), minimum (0, 0), and special elements e and ∂. Furthermore, $\leq$ is dense. This proves (I).

For condition (II), we need to define a new operation $\odot$ on Y, based on $\circ$, as follows:

$$(m,x)\odot(n,y) = \min\{\partial,(m,x)\circ(n,y)\} \text{ if } m=(\neg n)^+ \text{ and } p/q+p'/q' \leq 1,$$
$$\text{where } x = mp/q \text{ and } y=np'/q',$$
$$\text{or } m < (\neg n)^+; \text{ or}$$
$$\text{if } m=(\sim n)^+ \text{ and } p/q+p'/q' \leq 1,$$
$$\text{where } x = mp/q \text{ and } y=np'/q',$$

$$\text{or } m < (\sim n)^+; \text{ or}$$
$$(m,x) \circ (n,y) \quad \text{otherwise.}$$

The operation $\odot$ satisfies conditions (II) to (V) (see Proposition 2 in Yang (2017a)).

Now we note that, w.r.t. **IMIALc**$^r{}_n$f and **IMIALc**$^l{}_n$f, $3 \leq n$, the groupoid operation $\odot$ is defined as above, whereas w.r.t. **IMIAL**cf and **IMIAL**cpf (= **IMIALc**$^r{}_2$f, **IMIALc**$^l{}_2$f), the operation $\odot$ is defined based on the following definition of $\circ$: for (m, x), (n, y) $\in$ X,

$$
\begin{aligned}
(m,x) \circ (n,y) = {} & \max\{(m,x), (n,y)\} \text{ if } m * n =_A m \vee n \text{ and} \\
& \qquad (m, x) \succ e \text{ or } (n, y) \succ e; \\
& \min\{(m,x), (n,y)\} \text{ if } m * n = m \wedge z, \text{ and} \\
& \qquad (m, x) \preceq e \text{ or } (n, y) \preceq e; \\
& (m * n, m * n) \quad \text{otherwise.}
\end{aligned}
$$

The proof for each system is analogous to that for **IMIAL**. For **IMIAL**cf, we need to prove that (X, $\odot$, $\preceq$, e) satisfies ($c^{\mathbf{A}}$) and ($f^{\mathbf{A}}$). We first prove that $\odot$ satisfies ($c^{\mathbf{A}}$), i.e., (m, x) $\preceq$ (m, x) $\odot$ (m, x). Let m $\leq$ t. Since t = f, m < $(\neg m)^+$, $(\sim m)^+$ and thus (m, x) $\odot$ (m, x) = min{∂, (m, x) $\circ$ (m, x)}; therefore, (m, x) $\preceq$ (m, x) $\odot$ (m, x) since min{∂, (m, x) $\circ$ (m, x)} = (m, x) $\circ$ (m, x) and m = m * m. Let m > t. Since t = f, m > $(\neg m)^+$, $(\sim m)^+$ and thus (m, x) $\odot$ (m, x) = (m, x) $\circ$ (m, x); therefore, (m, x) $\preceq$ (m, x) $\odot$ (m, x) since (m, x) $\preceq$ (m, x) $\circ$ (m, x). We next prove that $\odot$ satisfies ($f^{\mathbf{A}}$), i.e., e = ∂. This directly follows from the fact that t = f and so (t, t) = (f, f).

For **IMIAL**cpf, we need to prove that (X, $\odot$, $\leq$, e) satisfies ($c^{\mathbf{A}}$), ($p^{\mathbf{A}}$), and ($f^{\mathbf{A}}$). We prove that $\odot$ satisfies ($p^{\mathbf{A}}$), i.e., (m, x) $\odot$ (m, x) $\leq$ (m, x). Let m $\leq$ t. Since t = f, m < $(\neg m)^+$, $(\sim m)^+$ and thus (m, x) $\odot$ (m, x) = min{∂, (m, x) ○ (m, x)}; therefore, (m, x) $\odot$ (m, x) $\leq$ (m, x) since min{∂, (m, x) ○ (m, x)} = (m, x) ○ (m, x) and m = m * m. Let m > t. Since t = f, m > $(\neg m)^+$, $(\sim m)^+$ and thus (m, x) $\odot$ (m, x) = (m, x) ○ (m, x); therefore, (m, x) $\odot$ (m, x) $\leq$ (m, x) since (m, x) $\leq$ (m, x) ○ (m, x) and m = m * m. For the conditions, ($c^{\mathbf{A}}$) and ($f^{\mathbf{A}}$), see the proof for **IMIAL**cf.

For **IMIAL**c^r_nf, we need to prove that (X, $\odot$, $\leq$, e) satisfies ($c^{r\,\mathbf{A}}_n$) and ($f^{\mathbf{A}}$). For this, see Proposition 3.2 in Yang (2017b).

The proof for **IMIAL**c^l_nf is analogous to that for **IMIAL**c^r_nf. □

Proposition 3.3 Every countable linearly ordered **IMIAL**cf-algebra (**IMIAL**c^r_nf-algebra, **IMIAL**c^l_nf-algebra, **IMIAL**cpf-algebra, respectively) can be embedded into a standard algebra.

Proof: In an analogy to the proof of Theorem 3.2 in Jenei & Montagna (2002), we prove this. Let X, **A**, etc. be as in Proposition 3.2. Since (X, $\leq$) is a countable, dense, linearly-ordered set with maximum and minimum, it is order isomorphic to ($\mathbf{Q}$ ∩ [0, 1], $\leq$). Let g be such an isomorphism. If (I) to (V) and the corresponding conditions hold, letting for α, β ∈ [0, 1], $\alpha \odot' \beta = g(g^{-1}(\alpha) \odot g^{-1}(\beta))$, and, for all m ∈ A,

$h'(m) = g(h(m))$, we obtain that $\mathbf{Q} \cap [0, 1]$, $\leq$, 1, 0, e, ∂, $\odot'$, h' satisfy the conditions (I) to (V) and additional conditions of Proposition 3.2 whenever X, $\preceq$, Max, Min, e, ∂, $\odot$, and h do. Thus, without loss of generality, we can assume that $X = \mathbf{Q} \cap [0, 1]$ and $\preceq\, = \,\leq$.

Now we define for $\alpha, \beta \in [0, 1]$,

$$\alpha \odot'' \beta = \sup_{x \in X: x \leq \alpha} \sup_{y \in X: y \leq \beta} x \odot y.$$

The monotonicity, identity, fixed-point, (and contraction, expansion, right n-potency and left n-potency, respectively) of $\odot''$ are easy consequences of the definition. Furthermore, it follows from the definition that $\odot''$ is conjunctive, i.e., $0 \odot'' 1 = 0$.

We prove left-continuity. Suppose that $<\alpha_n: n \in \mathbf{N}>$, $<\beta_n: n \in \mathbf{N}>$ are increasing sequences of reals in [0, 1] such that $\sup\{\alpha_n: n \in \mathbf{N}\} = \alpha$ and $\sup\{\beta_n: n \in \mathbf{N}\} = \beta$. By the monotonicity of $\odot''$, $\sup\{\alpha_n \odot'' \beta_n\} = \alpha \odot'' \beta$. Since the restriction of $\odot''$ to $\mathbf{Q} \cap [0, 1]$ is left-continuous, we obtain

$$\begin{aligned} \alpha \odot'' \beta &= \sup\{q \odot'' r: q, r \in \mathbf{Q} \cap [0, 1], q \leq \alpha, r \leq \beta\} \\ &= \sup\{q \odot'' r: q, r \in \mathbf{Q} \cap [0, 1], q < \alpha, r < \beta\}. \end{aligned}$$

For each $q < \alpha$, $r < \beta$, there is n such that $q < \alpha_n$ and $r < \beta_n$. Thus,

$$\sup\{\alpha_n \odot'' \beta_n: n \in \mathbf{N}\} \geq \sup\{q \odot'' r: q, r \in \mathbf{Q} \cap [0, 1], q < \alpha, r < \beta\} = \alpha \odot'' \beta.$$

Hence, $\odot''$ is a left-continuous involutive mianorm on [0, 1].

It is an easy consequence of the definition that $\odot''$ extends $\odot$. By (I) to (V) and the additional conditions of Proposition 3.2, h is an embedding of (A, $\leq_A$, $\top$, $\bot$, t, f, $\wedge$, $\vee$, *) into ([0, 1], $\leq$, 1, 0, e, ∂, min, max, $\odot''$). Moreover, $\odot''$ has a residuated pair of implications, calling it ($\rightharpoonup$, $\rightarrow$).

We finally prove that for x, y $\in$ A, h(x $\rightarrow$ y) = h(x) $\rightharpoonup$ h(y) and h(x $\Rightarrow$ y) = h(x) $\rightarrow$ h(y). By (IV), h(x $\rightarrow$ y) and h(x $\Rightarrow$ y) are the residuated pair of implications of h(x) and h(y) in (**Q** $\cap$ [0, 1], $\leq$, 1, 0, e, ∂, min, max, $\odot$). Thus

$$h(x) \odot'' h(x \rightarrow y) = h(x) \odot h(x \rightarrow y) \leq h(y), \text{ and}$$
$$h(x \Rightarrow y) \odot'' h(x) = h(x \Rightarrow y) \odot h(x) \leq h(y).$$

We prove the second one. Suppose toward contradiction that there is $\alpha > h(x \Rightarrow y)$ such that $\alpha \odot'' h(x) \leq h(y)$. Since **Q** $\cap$ [0, 1] is dense in [0, 1], there is q $\in$ **Q** $\cap$ [0, 1] such that $h(x \Rightarrow y) < q \leq \alpha$. Hence $q \odot'' h(x) = q \odot h(x) \leq h(y)$, contradicting the condition (IV). □

Theorem 3.4 (Strong standard completeness) For L ($\in$ Ls $\setminus$ {**IMIAL**cw, **IMIAL**c^r_nw, **IMIAL**c^l_nw}), the following are equivalent:

(1) T $\vdash_L$ ϕ.

(2) For every standard **L**-algebra and evaluation v, if $v(\psi) \geq e$ for all $\psi \in$ T, then $v(\phi) \geq e$.

Proof: (1) to (2) follows from the definition. We prove (2) to (1). Let ϕ be a formula such that $T \nvdash_L \phi$, **A** a linearly ordered **L**-algebra, and v an evaluation in **A** such that $v(\psi) \geq t$ for all $\psi \in T$ and $v(\phi) < t$. Let h′ be the embedding of **A** into the standard **L**-algebra as in proof of Proposition 3.3. Then, $h' \odot v$ is an evaluation into the standard **L**-algebra such that $h' \odot v(\psi) \geq e$ and yet $h' \odot v(\phi) < e$. □

Remark 3.5 (i) The proof of standard completeness in Theorem 3.4 does not work for the systems **IMIAL**cw, **IMIAL**c^r_nw, and **IMIAL**c^l_nw because the definition of $\odot$ does not satisfy contraction property. Consider the following case: $0 = \partial = \neg m < m = (\neg m)^+ = e = 1$ and $p/q + p/q \leq 1$, where $x = mp/q$. Since $m = m * m$, we have $(m, x) \odot (m, x) = \min\{\partial, (m, x) \circ (m, x)\} = \partial < (m, x)$; therefore, $(m, x) \not\leqslant (m, x) \odot (m, x)$.

(ii) In order to provide standard completeness results for the systems **IMIAL**cw, **IMIAL**c^r_nw, and **IMIAL**c^l_nw, one way to improve the Jenei-Montagna-style construction is to take the carrier set A having at least three elements in Propositions 3.2 and 3.3. Then, the counter-example in (i) does not hold since $0 = \partial = \neg m < (\neg m)^+ < m = e = 1$; therefore, $(m, x) \leqslant (m, x) \odot (m, x)$ since $(m, x) \odot (m, x) = (m, x) \circ (m, x)$.

4. Concluding remark

We investigated (not merely algebraic completeness for

IMIALcw, **IMIAL**cf, **IMIAL**c^r_nw, **IIMIAL**c^l_nw, **IMIAL**c^r_nf, **MIAL**c^l_nf, and **IMIAL**cpf but also) standard completeness for **IMIAL**cf, **IMIAL**c^r_nf, **IMIAL**c^l_nf, and **IMIAL**cpf. We further noted that in order to provide standard completeness for **IMIAL**cw, **I MIAL**c^r_nw, and **IMIAL**c^l_nw, we need some caution in taking a carrier set.

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전북대학교 철학과, 비판적사고와논술연구소
Department of Philosophy & Institute of Critical Thinking and Writing, Chonbuk National University
eunsyang@jbnu.ac.kr

누승적 미아놈 논리 **IMIAL**의 몇몇 공리적 확장

양 은 석

이 글에서 우리는 누승적 미아놈 논리 **IMIAL**의 몇몇 공리적 확장 체계의 표준 완전성을 다룬다. 이를 위하여, 먼저 누승적 미아놈에 바탕을 둔 일곱 개의 논리 체계를 소개한다. 각 체계에 상응하는 대수적 구조를 정의한 후, 이들 체계가 대수적으로 완전하다는 것을 보인다. 다음으로, 이 논리 체계들 중 네 체계가 표준적으로 완전하다는 것 즉 단위 실수 [0, 1]에서 완전하다는 것을 제네이-몬테그나 방식의 구성을 사용하여 보인다.

주요어: 퍼지 논리, 누승, 미아놈, 대수적 완전성, 표준 완전성.