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Liar-Type Paradoxes and Intuitionistic Natural Deduction Systems*

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【Abstract】It is often said that in a purely formal perspective, intuitionistic logic has no obvious advantage to deal with the liar-type paradoxes. In this paper, we will argue that the standard intuitionistic natural deduction systems are vulnerable to the liar-type paradoxes in the sense that the acceptance of the liar-type sentences results in inference to absurdity ($\bot$). The result shows that the restriction of the *Double Negation Elimination* (*DNE*) fails to block the inference to $\bot$. It is, however, not the problem of the intuitionistic approaches to the liar-type paradoxes but the lack of expressive power of the standard intuitionistic natural deduction system.

We introduce a meta-level negation, $\nvdash_S$, for a given system S and a meta-level absurdity, $\curlywedge$, to the intuitionistic system. We shall show that in the system, the inference to $\bot$ is not given without the assumption that the system is complete. Moreover, we consider the *Double Meta-Level Negation Elimination* rules (*DMNE*) which implicitly assume the completeness of the system. Then, the restriction of *DMNE* can rule out the inference to $\bot$.

【Key Words】Liar paradox, Strengthened liar paradox, Revenge liar, Natural deduction, Double negation elimination, Intuitionistic logic.

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1 Introduction

Richard M. Sainsbury (2009, p.1) defines a paradox as 'an apparently unacceptable conclusion derived by apparently acceptable reasoning from apparently acceptable premises.' As the acceptable processes hardly draw the unacceptable conclusion from the acceptable premises, there are three ways to solve the paradox. At first, one can reject that the premises are acceptable. The second is to reject the acceptability of the reasoning. The last is to claim that the conclusion is acceptable.

'This sentence is not true' is a well-known liar sentence which leads to the liar paradox. The liar sentence is equivalent to its negation in virtue of its meaning. The derivation of absurdity ($\perp$) in the case of the liar paradox is given by making use of the *Law of Excluded Middle* (*LEM*). (See Appendix B.) *LEM* is often regarded as the source of the derivation of $\perp$ from the liar-type paradoxes. (Cf. Hartry Field (2008, p. 15).) *LEM* seems to be a requisite inference for deriving $\perp$ from the liar sentence.

The main theme of intuitionism is to reject the *Principle of Bivalence* (*POB*) that every sentence is either true or false. Even if the rejection of *POB* does not always imply the rejection of *LEM*, an intuitionist welcomes to refuse both *POB* and *LEM*.[1] An intuitionistic interpretation of truth gives three cases of determining the truth-value of a given sentence: true, false, and truth-valueless. A sentence φ is true if it has a proof, and is false if it has a disproof; or else it is truth-valueless. When the liar sentence has no proof, nothing can transfer any proof of the liar sentence to $\perp$, so the liar sentence

[1] A system with the decidable set of sentences can accept *LEM* without the assumption of *POB*.

would have no consequence. Hence, there is no inference to $\bot$ from the liar sentence, whereas the proof in Appendix B draws the inference to $\bot$ with the application of *LEM*. In this sense, an intuitionistic solution to the liar paradox is to claim that the inference to $\bot$ from the liar sentence has an unacceptable rule of reasoning, such as *LEM*. Hence, intuitionistic logic has the advantage to block the liar paradox by rejecting *POB* and *LEM*.

The intuitionistic solution has been challenged by the view that there exists an inference to $\bot$ without *POB* and *LEM*. Graham Priest (1983, 2006) proposes a proof of $\bot$ from Berry's paradox without *LEM*. If his claims were correct, there exists a case that *POB* and *LEM* play no role at all in the inference to $\bot$. Furthermore, philosophers have claimed that, in a purely technical sense, an intuitionistic formal system cannot rule out the inferences of $\bot$. For example, when he argues that semantic paradoxes are not linked to Michael Dummett's notion of the indefinitely extensible concepts, Timothy Williamson (1998, p. 2) mentions a passing remark that intuitionistic logic has no advantage to block paradoxes, by saying,

> From a purely technical perspective, intuitionistic logic presents no obvious advantage. In the simplest paradoxes, a plausible general principle turns out to have a substitution instance of the form $[\varphi \leftrightarrow \neg\varphi]$, which is inconsistent in both intuitionistic and classical logic. Adopting intuitionistic logic would not enable us to retain the plausible general principle while blocking the inference to a contradiction.[2]

[2] An indefinitely extensible concept is one whose extension cannot be completely determined: any set of objects that falls under the relevant concept can be extended. Dummett (1993, p. 454) claims that paradoxes are generated by possessing indefinitely extensible concepts. It is often noted that Dummett's claim is extended to an argument against classical logic. The logic in which indefinitely extensible concepts are expressible is non-classical because all statements in it are not determinately true

Also, Neil Tennant (2017, p. 284) has maintained that *LEM* does not have any role in the derivation of $\perp$ from paradoxes, so paradoxical inferences to $\perp$ can be derived by using only intuitionistic logic. The present paper deals with these challenges and queries on whether intuitionistic logic is in practice vulnerable to the liar-type paradoxes.

An intuitionistic natural deduction system styled by Dag Prawitz (1965) is a primary candidate formal system for intuitionistic logic. We firstly show that in the intuitionistic natural deduction systems, there exist derivations of $\perp$ from the (strengthened) liar paradox without the application of the *Double Negation Elimination* (*DNE*) which is provably equivalent to *LEM*. In addition, it can be seen that a liar-type sentence gives a particular instance of *DNE*. It underscores the fact that the liar-type sentence has its proof in the intuitionistic natural deduction system. If the intuitionistic solution to the liar-type paradoxes claims that the liar-type sentences have no proof-condition, the suggested results fall foul of the intuitionistic solution.

In what follows, we will investigate the case that the restriction of any instances of *DNE* and of *DNE* itself fail to rule out the derivation of $\perp$ in the suggested system. Even though the situation seems to support the above challenge, it does not count against the intuitionistic solution to the liar-type paradoxes. Rather, it is an incapability of the formal system to characterize the intuitionistic meaning of negation. We will reinforce the intuitionistic system by applying a meta-level negation, $\nvdash_S$, for a given system S and a meta-level absurdity, $\curlywedge$, to the system. We shall show that the inference to $\perp$ is not derivable if the system is not complete. Moreover, we will consider $\curlywedge_C-$rules for the *Double Meta-Level Negation Elimination* (*DMNE*).

or false. Dummett's indefinitely extensible concept is not an issue of this paper, so we leave aside the issue.

With $\curlywedge_C$−rules, we have a derivation of $\bot$ in a given system since $\curlywedge_C$−rules implicitly assume that the system is complete. $\curlywedge_C$−rules play a significant role to derive $\bot$.

Section 2 briefly introduces preliminary notations and logical rules. Section 3 provides a proof of $\bot$ from the liar sentence in the intuitionistic natural deduction system, S_{IT}, having the truth-predicate and gives an instance of *DNE* from the liar sentence. In Section 4, we will investigate the strengthened liar paradox analyzed in S_{ITU} with the predicate of the truth-valuelessness, say $U_{S_{ITU}}(\ulcorner x \urcorner)$. Section 5 argues that the restriction of *DNE* does not block the inferences to $\bot$ in S_{IT} and S_{ITU}. The discussions from Section 3 to 5 appear to support that an intuitionistic natural deduction system does not have a clear benefit to block the inference to $\bot$ from the liar-type paradoxes. We will argue in Section 6, however, that the problem is the lack of expressive power of the system, but not the intuitionistic approaches to the liar-type paradoxes. We shall have an intuitionistic system $S_{I'}$ by applying the meta-level negation, $\nvdash_{S_{I'}}$, and absurdity, $\curlywedge$, to the intuitionistic system S_I, and show that there exists no inference to $\bot$ unless $S_{I'}$ is complete. Consequenty, even if the restriction of *DNE* fails to block the inference to $\bot$, the restriction of *DMNE* prevents the inference to $\bot$ in $S_{I'}$.

2 The Intuitionistic Interpretation and the Natural Deduction System S_I

A natural deduction system may be well-suited to the intuitionistic interpretation of logical constants. Following Gerhard Gentzen (1935), Prawitz (1965) gives a reference work on the natural deduction system with the idea that the meaning of logical constants is implicitly defined by its introduction rules ($I-$rules), while the elimination rules (E-rules) are justified by respecting the stipulation made by the $I-$rules. As Prawitz (2016) has argued that the idea is extensionally equivalent to the intuitionistic interpretation, an intuitionistic natural deduction system may be an appropriate formal system for intuitionistic logic. In this section, we briefly introduce a standard intuitionistic interpretation of logical constants with some variations for our purpose. Also, we give logical rules for an intuitionistic system S_I in the natural deduction style proposed by Prawitz (1965).

Let $\mathfrak{L}$ be a language of intuitionistic logic and $\mathbb{P}$ be a set of proofs of atomic sentences of $\mathfrak{L}$. $\mathbb{O}$ be a set of objects o. Say x, y be any free variables and t be any term not free. φ, ψ, χ be any formulas. We assume, for convenience, that each object o in $\mathbb{O}$ has its term t. For any formula $\varphi(x)$, we understand by $\varphi(t)$ the closed sentence obtained by substituting in $\varphi(x)$ the term t of o for x. Let $\mathfrak{D}$ be a sequence of (constructive) proofs, say 'derivation,' used in the same manner as 'deduction' in Prawitz (1965, p. 24). We use $\begin{array}{c}\mathfrak{D}\\ \varphi\end{array}$ for a derivation for φ – i.e. a sequence of the proofs of φ. $\begin{array}{c}\varphi\\ \mathfrak{D}\\ \psi\end{array}$ means a derivation of ψ from φ. Let $\mathfrak{D}_1, ..., \mathfrak{D}_n$ be an arbitrary derivation with respect to $\mathbb{P}$. The intuitionistic interpretation through a proof over $\mathbb{P}$ of a closed compound sentence in $\mathfrak{L}$ is then proposed recursively as follows:

For any sentence φ_1 and φ_2,

1. A proof over $\mathbb{P}$ of $\varphi_1 \wedge \varphi_2$ is a pair $(\mathfrak{D}_1, \mathfrak{D}_2)$ of derivations such that $\begin{array}{c}\mathfrak{D}_1\\ \varphi_1\end{array}$ and $\begin{array}{c}\mathfrak{D}_2\\ \varphi_2\end{array}$.

2. A proof over $\mathbb{P}$ of $\varphi_1 \vee \varphi_2$ is a pair $(\mathfrak{D}_1, \mathfrak{D}_2)$ of derivations such that $\begin{matrix}\mathfrak{D}_1\\ \varphi_1\end{matrix}$ or $\begin{matrix}\mathfrak{D}_2\\ \varphi_2\end{matrix}$.
3. A proof over $\mathbb{P}$ of $\varphi \rightarrow \psi$ is a derivation $\mathfrak{D}_1$ which converts any proof of φ into a proof of ψ. (i.e. $\begin{matrix}\varphi\\ \mathfrak{D}\\ \psi\end{matrix}$)
4. Nothing is a proof of $\perp$.
5. A proof over $\mathbb{P}$ of $\exists x\varphi(x)$ is a derivation $\mathfrak{D}_1$ such that $\begin{matrix}\mathfrak{D}_1\\ \varphi(t)\end{matrix}$ where o of t in $\mathbb{O}$.
6. A proof over $\mathbb{P}$ of $\forall x\varphi(x)$ is a derivation $\mathfrak{D}_1$ which converts any object o of $\mathbb{O}$ into a proof of $\varphi(y)$ where y of any o in $\mathbb{O}$.

A classical natural deduction system S_C has the following rules:

$$\dfrac{\begin{matrix}\mathfrak{D}_1\\ \varphi_1\end{matrix} \quad \begin{matrix}\mathfrak{D}_2\\ \varphi_2\end{matrix}}{\varphi_1 \wedge \varphi_2}\wedge I \qquad \dfrac{\begin{matrix}\mathfrak{D}_3\\ \varphi_1 \wedge \varphi_2\end{matrix}}{\varphi_i}\wedge E_{i=1,2} \qquad \dfrac{\begin{matrix}[\varphi]^1\\ \mathfrak{D}_1\\ \psi\end{matrix}}{\varphi \rightarrow \psi}\rightarrow I_{,1} \qquad \dfrac{\begin{matrix}\mathfrak{D}_2\\ \varphi \rightarrow \psi\end{matrix} \quad \begin{matrix}\mathfrak{D}_3\\ \varphi\end{matrix}}{\psi}\rightarrow E$$

$$\dfrac{\begin{matrix}\mathfrak{D}_1\\ \varphi_i\end{matrix}}{\varphi_1 \vee \varphi_2}\vee I_{(i=1,2)} \qquad \dfrac{\begin{matrix}\mathfrak{D}_2\\ \varphi_1 \vee \varphi_2\end{matrix} \quad \begin{matrix}[\varphi_1]^1\\ \mathfrak{D}_3\\ \psi\end{matrix} \quad \begin{matrix}[\varphi_2]^1\\ \mathfrak{D}_4\\ \psi\end{matrix}}{\psi}\vee E_{,1} \qquad \dfrac{\begin{matrix}\mathfrak{D}\\ \perp\end{matrix}}{\varphi}\perp_I \qquad \dfrac{\begin{matrix}[\neg\varphi]^1\\ \mathfrak{D}\\ \perp\end{matrix}}{\varphi}\perp_{C,1}$$

$$\dfrac{\begin{matrix}\mathfrak{D}_1\\ \varphi[y/x]\end{matrix}}{\forall x\varphi(x)}\forall I \qquad \dfrac{\begin{matrix}\mathfrak{D}_2\\ \forall x\varphi(x)\end{matrix}}{\varphi[t/x]}\forall E \qquad \dfrac{\begin{matrix}\mathfrak{D}_1\\ \varphi[t/x]\end{matrix}}{\exists x\varphi(x)}\exists I \qquad \dfrac{\begin{matrix}\mathfrak{D}_2\\ \exists x\varphi(x)\end{matrix} \quad \begin{matrix}[\varphi[y/x]]^1\\ \mathfrak{D}_3\\ \psi\end{matrix}}{\psi}\exists E_{,1}$$

$\neg\varphi$ is defined by $\varphi \rightarrow \perp$. $\varphi[x/y]$ means the substitution of x for y in φ. A restriction of the eigenvariable y in $\forall I-$ and $\exists E-$rules runs in a usual way. A derivation is *open* when it depends on assumptions and *closed* when all assumptions are discharged or bound. *DNE* has a form of the formula $\neg\neg\varphi \to \varphi$ and its relative rule is $\bot_C-$rule. An intuitionistic natural deduction system S_I is given by dropping $\bot_C-$rule from S_C. In addition, a minimal system S_M is taken by dropping $\bot_I$-rule from S_I. We write '$S_I \vdash \varphi$' to mean that S_I derives φ and '$S_I \nvdash \varphi$' means that S_I does not.[3] We say that a system is *intuitionistic* if it extends S_I and any classical rules, e.g. $\bot_C-$, $\curlywedge_C-$rules, and *LEM*, are not always admissible.[4] Hence, if S is intuitionistic, then, for some φ in $\mathfrak{L}$, $S \nvdash \neg\neg\varphi \to \varphi$.

Having this conception of the intuitionistic interpretation with its natural deduction system, we will see in the next section that there exists a derivation of $\bot$ without the application of $\bot_C-$rule and the liar sentence Φ implies $\neg\neg\Phi \to \Phi$.

3 The Liar Paradox and *DNE*

Let us consider that a language $\mathfrak{L}$ of S_I has a truth predicate $T(\ulcorner x \urcorner)$ where '$\ulcorner$' and '$\urcorner$' are the left and the right corner quotes. $T(\ulcorner x \urcorner)$ expresses that $\ulcorner x \urcorner$ is true where $\ulcorner x \urcorner$ denotes x. Then, we have the

[3] Note that '$S \nvdash \varphi$' and '$\nvdash_S \varphi$' are separated in this paper. '$\nvdash$' is a binary predicate and '$\nvdash_S$' is a meta-level negation for a system S. They express the same meaning that S does not derive φ, but we shall not have any rule to derive $\curlywedge$ from $S \nvdash \varphi$ and φ, save from $\nvdash_S \varphi$ and φ where $\nvdash_S \varphi$ is defined by $\varphi \to \curlywedge$ in a given system S. (Cf. Section 4 and 6.)

[4] Let R be a rule with premises $\varphi_1, \dots, \varphi_n$ and a conclusion ψ. S be a system of rules. R is said to be *admissible* for S if $S \vdash \varphi_1, \dots, \varphi_n$ implies $S \vdash \psi$.

following rules for $T(\ulcorner x \urcorner)$.

$$\dfrac{\begin{array}{c}\mathfrak{D}_1\\ \varphi\end{array}}{T(\ulcorner \varphi \urcorner)}\,TI \qquad \dfrac{\begin{array}{c}\mathfrak{D}_2\\ T(\ulcorner \varphi \urcorner)\end{array}}{\varphi}\,TE$$

A natural deduction system S_{IT} is given by adding $TI-$ and $TE-$rules to S_I. $\varphi \leftrightarrow \psi$ is defined by $(\varphi \to \psi) \wedge (\psi \to \varphi)$. We define a formula Φ as $\neg T(\ulcorner \Phi \urcorner)$. Then, we have a relation $\Phi \leftrightarrow \neg T(\ulcorner \Phi \urcorner)$. We say that φ is a *liar* sentence if φ satisfies the relation $\varphi \leftrightarrow \neg T(\ulcorner \varphi \urcorner)$. After having additional terminologies, we will prove that, for a liar sentence Φ, if $S_{IT} \vdash \Phi \leftrightarrow \neg T(\ulcorner \Phi \urcorner)$, then $S_{IT} \vdash \perp$. Provided that S_{IT} accepts $\Phi \leftrightarrow \neg T(\ulcorner \Phi \urcorner)$ as a theorem, the result shows that there is a closed derivation of $\perp$ in S_{IT}.

Suppose that there is a liar sentence Φ in $\mathfrak{L}$ of S_{IT} and $S_{IT} \vdash \Phi \leftrightarrow \neg T(\ulcorner \Phi \urcorner)$. It means that S_{IT} has a closed derivation $\mathfrak{D}_{\mathfrak{L}_1}$ of $\Phi \leftrightarrow \neg T(\ulcorner \Phi \urcorner)$. For our convenience sake, we use the following abbreviations:

$$\dfrac{\begin{array}{c}\mathfrak{D}\\ \Phi\end{array}}{\neg T(\ulcorner \Phi \urcorner)}\,LI_1$$

is an abbreviation for

$$\dfrac{\begin{array}{c}\mathfrak{D}\\ \Phi\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_{\mathfrak{L}_1}\\ \Phi \leftrightarrow \neg T(\ulcorner \Phi \urcorner)\end{array}}{(\Phi \to \neg T(\ulcorner \Phi \urcorner)) \wedge (\neg T(\ulcorner \Phi \urcorner) \to \Phi)}\,def}{\Phi \to \neg T(\ulcorner \Phi \urcorner)}\,\wedge E}{\neg T(\ulcorner \Phi \urcorner)}\,\to E$$

Then,

$$\dfrac{\begin{array}{c}\mathfrak{D}\\ \neg T(\ulcorner\Phi\urcorner)\end{array}}{\Phi}\,LE_1$$

is an abbreviation for

$$\dfrac{\begin{array}{c}\mathfrak{D}\\ \neg T(\ulcorner\Phi\urcorner)\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_{\mathfrak{L}_1}\\ \Phi\leftrightarrow\neg T(\ulcorner\Phi\urcorner)\end{array}}{(\Phi\to\neg T(\ulcorner\Phi\urcorner)\wedge(\neg T(\ulcorner\Phi\urcorner)\to\Phi)}\,def}{\neg T(\ulcorner\Phi\urcorner)\to\Phi}\,\wedge E}{\Phi}\,\to E$$

$\neg\Phi \leftrightarrow T(\ulcorner\Phi\urcorner)$ is provable in S_{IT}. (See Appendix A). There is a derivation $\mathfrak{D}_{\mathfrak{L}_2}$ of $\neg\Phi\leftrightarrow T(\ulcorner\Phi\urcorner)$. Then,

$$\dfrac{\begin{array}{c}\mathfrak{D}\\ \neg\Phi\end{array}}{T(\ulcorner\Phi\urcorner)}\,LI_2$$

is an abbreviation for

$$\dfrac{\begin{array}{c}\mathfrak{D}\\ \neg\Phi\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_{\mathfrak{L}_2}\\ \neg\Phi\leftrightarrow T(\ulcorner\Phi\urcorner)\end{array}}{(\neg\Phi\to T(\ulcorner\Phi\urcorner))\wedge(T(\ulcorner\Phi\urcorner)\to\neg\Phi)}\,def}{\neg\Phi\to T(\ulcorner\Phi\urcorner)}\,\wedge E}{T(\ulcorner\Phi\urcorner)}\,\to E$$

In addition,

$$\dfrac{\begin{array}{c}\mathfrak{D}\\ T(\ulcorner\Phi\urcorner)\end{array}}{\neg\Phi}\,LE_2$$

is an abbreviation for

$$\dfrac{\begin{matrix}\mathfrak{D}\\ T(\ulcorner\Phi\urcorner)\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\mathfrak{D}_{\mathfrak{L}_2}\\ \neg\Phi\leftrightarrow T(\ulcorner\Phi\urcorner)\end{matrix}}{(\neg\Phi\to T(\ulcorner\Phi\urcorner)\wedge(T(\ulcorner\Phi\urcorner)\to\neg\Phi)}\,def}{T(\ulcorner\Phi\urcorner)\to\neg\Phi}\,\wedge E}{\neg\Phi}\to E$$

For a liar sentence Φ, if S_{IT} has a closed derivation $\mathfrak{D}$ of Φ, then we easily prove that S_{IT} has a closed derivation of $\bot$ in S_{IT}.

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}\\ \Phi\end{matrix}}{T(\ulcorner\Phi\urcorner)}\,TI \qquad \dfrac{\begin{matrix}\mathfrak{D}\\ \Phi\end{matrix}}{\neg T(\ulcorner\Phi\urcorner)}\,LI_1}{\bot}\to E$$

However, it is unclear whether S_{IT} has a closed derivation of Φ. Rather one may claim that $\Phi\leftrightarrow\neg T(\ulcorner\Phi\urcorner)$ is true because Φ is defined by $\neg T(\ulcorner\Phi\urcorner)$. If so, we have the following results.

Theorem 3.1. *If $S_{IT}\vdash\Phi\leftrightarrow\neg T(\ulcorner\Phi\urcorner)$, then $S_{IT}\vdash\Phi\leftrightarrow\neg\Phi$.*

Proof. Suppose that $S_{IT}\vdash\Phi\leftrightarrow\neg T(\ulcorner\Phi\urcorner)$. Then, we can use LI_1-,LI_2-, LE_1-, and LE_2-inferences.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\Phi]^1}{T(\ulcorner\Phi\urcorner)}\,TI}{\neg\Phi}\,LE_2}{\Phi\to\neg\Phi}\to I_{,1} \qquad \dfrac{\dfrac{\dfrac{[\neg\Phi]^2}{T(\ulcorner\Phi\urcorner)}\,LI_2}{\Phi}\,TE}{\neg\Phi\to\Phi}\to I_{,2}}{(\Phi\to\neg\Phi)\wedge(\neg\Phi\to\Phi)}\,\wedge I}{\Phi\leftrightarrow\neg\Phi}\,def$$

Corollary 3.2. *If $S_{IT} \vdash \Phi \leftrightarrow \neg T(\ulcorner\Phi\urcorner)$, then $S_{IT} \vdash \bot$.*

Proof. By Theorem 3.1, we have $S_{IT} \vdash \Phi \leftrightarrow \neg\Phi$. Then we have a derivation $\begin{matrix}\mathfrak{D}_1\\ \neg\Phi\end{matrix}$.

$$\dfrac{[\Phi]^1 \quad \dfrac{[\Phi]^1 \quad \dfrac{\dfrac{\overline{\Phi \leftrightarrow \neg\Phi}\ \text{theorem 3.1}}{(\Phi \to \neg\Phi) \wedge (\neg\Phi \to \Phi)}\ def}{\Phi \to \neg\Phi}\ \wedge E}{\neg\Phi}\ \to E}{\dfrac{\bot}{\neg\Phi}\ \to I_{,1}}\ \to E$$

With the derivation $\mathfrak{D}_1$, we either have a derivation $\mathfrak{D}_2$ of Φ.

$$\dfrac{\begin{matrix}\mathfrak{D}_1\\ \neg\Phi\end{matrix} \quad \dfrac{\dfrac{\overline{\Phi \leftrightarrow \neg\Phi}\ \text{theorem 3.1}}{(\Phi \to \neg\Phi) \wedge (\neg\Phi \to \Phi)}\ def}{\neg\Phi \to \Phi}\ \wedge E}{\Phi}\ \to E$$

Hence, the following process shows that $S_{IT} \vdash \bot$.

$$\dfrac{\begin{matrix}\mathfrak{D}_1\\ \neg\Phi\end{matrix} \quad \begin{matrix}\mathfrak{D}_2\\ \Phi\end{matrix}}{\bot}\ \to E$$

Furthermore, in the proof of Corollary 3.2, $\Phi \leftrightarrow \neg T(\ulcorner\Phi\urcorner)$ leads to a derivation $\mathfrak{D}_2$ of Φ. In this respect, we have the following result.

Corollary 3.3. *If* $S_{IT} \vdash \Phi \leftrightarrow \neg T(\ulcorner \Phi \urcorner)$, *then* $S_{IT} \vdash \neg\neg\Phi \rightarrow \Phi$.

Proof. From the derivation $\mathfrak{D}_2$ of Φ in Corollary 3.2, we apply $\rightarrow I$−rule with the empty discharge and have $\neg\neg\Phi \rightarrow \Phi$. □

When every sentence is derivable in a system, we say that it is *trivial*. The proof of Corollary 3.3 does not use $\bot_I$−rule and is provable in a weaker system S_M with the rules for $T(\ulcorner x \urcorner)$. The triviality of S_{IT} does not matter for the proof of $\neg\neg\Phi \rightarrow \Phi$. Corollary 3.3 tells us that the acceptance of $\Phi \leftrightarrow \neg T(\ulcorner \Phi \urcorner)$ in S_{IT} admits the classical inference, such as *DNE* (or $\bot_C$−rule). An intuitionist accepts $\bot_C$−rule for a formula that has a (constructive) proof. The result claims that an intuitionistic natural deduction system S_{IT} represents no more obvious advantage than a classical system.

The intuitionist may answer that, for a liar sentence Φ, $\Phi \leftrightarrow \neg T(\ulcorner \Phi \urcorner)$ is not intuitionistically true, due to the fact that neither the proof-condition of Φ nor $\Phi \leftrightarrow \neg T(\ulcorner \Phi \urcorner)$ is informed. For instance, if $\neg T(\ulcorner \Phi \urcorner)$ means the truth-valuelessness of Φ and has no proof, nothing can transfer any proof of $\neg T(\ulcorner \Phi \urcorner)$ to a proof of Φ. Namely, $\neg T(\ulcorner \Phi \urcorner) \rightarrow \Phi$ is not intuitionistically true. Hence, while the proof-condition of the liar-sentence is not known, $\Phi \leftrightarrow \neg T(\ulcorner \Phi \urcorner)$ is not intuitionistically true.

On the other hand, some philosophers, like Priest (2006), can claim that the inference to $\bot$ occurs from the liar-type paradoxes even though the truth-valuelessness is concerned. In the next section, we will see a derivation of $\bot$ from the strengthened liar paradox.

4 The Intuitionistic 'not true' and The Strengthened Liar Paradox

An intuitionist interprets the concept of truth in terms of a constructive proof such that a sentence φ is (intuitionistically) true if and only if φ has a constructive proof. In a classical interpretation, the sentence 'φ is not true' has the same meaning of 'φ is false,' whereas, in an intuitionistic interpretation, the former is much weaker than the latter. Intuitionistically, 'φ is false,' that is '$\neg\varphi$ is true,' implies 'φ is not true,' but not vice versa. Thus, 'φ is not true in a system S' either means $S \vdash \neg\varphi$ or $S \nvdash \varphi$.

As has often been discussed, the liar sentence Φ will yield no determinate truth-(or proof-)condition. The intuitionist will claim that none of Φ and $\neg\Phi$ have proofs, which means that they are truth-valueless. On the intuitionistic interpretation of $\rightarrow$, there is no effective operation transforming any proof of Φ (or $\neg\Phi$) into the conclusion, so $\Phi \leftrightarrow \neg T(\ulcorner\Phi\urcorner)$ has no proof. Therefore, the inference from $\Phi \leftrightarrow \neg T(\ulcorner\Phi\urcorner)$ to $\bot$ does not occur.[5] The strengthened liar paradox is suggested at this point.

Let us consider a sentence Ψ in $\mathfrak{L}$ which expresses that 'Ψ' is false or truth-valueless. We suppose that every sentence is either true, false, or truth-valueless, and that no sentence is more than one of true, false, or truth-valueless. We say any form of the sentence $\varphi \wedge \neg\varphi$ contradiction. At first, if Ψ is true, then it is false or truth-valueless. Hence it is either both true and false or both true and truth-valueless.

[5]One may deal with the sentence $\Phi \leftrightarrow \neg T(\ulcorner\Phi\urcorner)$ in the same way as $\bot$ since $\bot$ has no proof. In this case, there exists an inference from $\Phi \leftrightarrow \neg T(\ulcorner\Phi\urcorner)$ to $\bot$, i.e. an absurd to an absurd. If $\Phi \leftrightarrow \neg T(\ulcorner\Phi\urcorner)$ has no proof and is absurd, it trivially implies $\bot$. It may be a candidate interpretation of the inference to $\bot$ from $\Phi \leftrightarrow \neg T(\ulcorner\Phi\urcorner)$, but in this paper, we set aside the interpretation.

This contradicts to the supposition, so Ψ is not true. Second, if Ψ is false, then it is either false or truth-valueless. Ψ says that Ψ is either false or truth-valueless. So what Ψ says is the case. Thus, Ψ is true and false. Once again, a contradiction occurs, so it is not false. The last option is to assume that Ψ is truth-valueless. However, the similar inference leads us to a contradiction. From all the cases of the strengthened liar paradox, we derive a contradiction (or absurdity).

To realize the strengthened liar paradox in a natural deduction system, the truth-valuelessness must be expressible in the system. As the claim 1 of Appendix A shows, the meaning of $\neg T(\ulcorner\varphi\urcorner)$ is the same as of $T(\ulcorner\neg\varphi\urcorner)$. $\neg T(\ulcorner\varphi\urcorner)$ in the system does not express the truth-valuelessness of φ, but the falsehood of φ. For a given system S and a sentence φ in $\mathfrak{L}$, if $S \nvdash \varphi$ and $S \nvdash \neg\varphi$, then φ and $\neg\varphi$ have no proof (or derivation) in S. The truth-valuelessness of φ in S_{IT} is represented by the underivability, such as $S_{IT} \nvdash \varphi$ and $S_{IT} \nvdash \neg\varphi$. Unfortunately, S_{IT} is incapable of representing the underivability because it does not have any rule to derive $S_{IT} \nvdash \varphi$ (or $S_{IT} \nvdash \neg\varphi$).

We assume that the underivability is expressible in the language of an intuitionistic system. For any given intuitionistic system S, we define a meta-level negation for S, $\nvdash_S \varphi$, as $\varphi \to \curlywedge$ where '$\nvdash_S \varphi$' means $S \nvdash \varphi$ and $\curlywedge$ is a meta-level absurdity constant. For any system S, S is *incomplete* if, for some φ in $\mathfrak{L}$, $S \nvdash \varphi$ and $S \nvdash \neg\varphi$; otherwise complete. $\bot$ and $\curlywedge$ are distinguished in an incomplete system. For instance, if S_{IT} has both derivations of φ and $\nvdash_{S_{IT}} \varphi$, then by $\to E$−rule, we have $S_{IT} \vdash \curlywedge$. However, $\curlywedge$ does not always mean $\bot$ since $S_{IT} \nvdash \varphi$ does not imply $S_{IT} \vdash \neg\varphi$ unless S_{IT} is complete. Hence, $\bot$ is derivable in S_{IT} from φ and $\nvdash_{S_{IT}} \varphi$ only if S_{IT} is complete.[6]

[6] An anonymous reviewer and Colin Caret say that it should be discussed whether the meta-level negation and absurdity are philosophically acceptable. Truly, there is

For a given system S and a sentence φ, we define $U_S(\ulcorner\varphi\urcorner)$ as the sentence $\nvdash_S \varphi \wedge \nvdash_S \neg\varphi$. We have a natural deduction system S_{ITU} by adding the predicate $U_{S_{ITU}}(\ulcorner x\urcorner)$ to S_{IT}. As noted above, in S_{IT} and S_{ITU}, $\neg T(\ulcorner\varphi\urcorner) \leftrightarrow T(\ulcorner\neg\varphi\urcorner)$ is provable, $\neg T(\ulcorner\varphi\urcorner)$ expresses that φ is false. Let define Ψ as $\neg T(\ulcorner\Psi\urcorner) \vee U_{S_{ITU}}(\ulcorner\Psi\urcorner)$. Then, for any given sentence φ and any system S, φ is a *strengthened liar* sentence over S if φ satisfies the relation $\varphi \leftrightarrow \neg T(\ulcorner\varphi\urcorner) \vee U_S(\ulcorner\varphi\urcorner)$.

Suppose that there exists a strengthened liar sentence Ψ over S_{ITU} in $\mathfrak{L}$ and S_{ITU} has a derivation $\mathfrak{D}_{\mathfrak{sl}}$ of $\Psi \leftrightarrow \neg T(\ulcorner\Psi\urcorner) \vee U_{S_{ITU}}(\ulcorner\Psi\urcorner)$. For convenience, we use the following abbreviations:

$$\dfrac{\begin{matrix}\mathfrak{D}\\ \Psi\end{matrix}}{\neg T(\ulcorner\Psi\urcorner) \vee U_{S_{ITU}}(\ulcorner\Psi\urcorner)}\,SLI$$

is an abbreviation for

$$\dfrac{\begin{matrix}\mathfrak{D}\\ \Psi\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\mathfrak{D}_{\mathfrak{sl}}\\ \Psi \leftrightarrow \neg T(\ulcorner\Psi\urcorner) \vee U_{S_{ITU}}(\ulcorner\Psi\urcorner)\end{matrix}}{(\Psi \to \neg T(\ulcorner\Psi\urcorner) \vee U_{S_{ITU}}(\ulcorner\Psi\urcorner)) \wedge (\neg T(\ulcorner\Psi\urcorner) \vee U_{S_{ITU}}(\ulcorner\Psi\urcorner) \to \Psi)}\,def}{\Psi \to \neg T(\ulcorner\Psi\urcorner) \vee U_{S_{ITU}}(\ulcorner\Psi\urcorner)}\,\wedge E}{\neg T(\ulcorner\Psi\urcorner) \vee U_{S_{ITU}}(\ulcorner\Psi\urcorner)}\,\to E$$

no consensus of the interpretation of intuitionistic negation and absurdity. Prawitz (2007, p.461) regards $\bot$ as an atomic sentence which expresses falsehood, but Dummett (1991, pp.294-295) defines $\bot$ as the conjunction of all atomic sentences. Moreover, Tennant (1999) considers $\bot$ to be nothingness in the same way that the ancient Hindus used ‘0’ for emptiness in arithmetic. In this regard, we may think that the meta-level absurdity, $\curlywedge$, means nothing and $\bot$ stands for falsehood. Unfortunately, space is insufficient to allow a more detailed discussion of this issue. We set aside the issue in this paper.

Also,

$$\dfrac{\begin{matrix}\mathfrak{D}\\ \neg T(\ulcorner\Psi\urcorner)\vee U_{S_{ITU}}(\ulcorner\Psi\urcorner)\end{matrix}}{\Psi}\ SLE$$

is an abbreviation for

$$\dfrac{\begin{matrix}\mathfrak{D}\\ \neg T(\ulcorner\Psi\urcorner)\vee U_{S_{ITU}}(\ulcorner\Psi\urcorner)\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\mathfrak{D}_{\mathfrak{sl}}\\ \Psi\leftrightarrow\neg T(\ulcorner\Psi\urcorner)\vee U_{S_{ITU}}(\ulcorner\Psi\urcorner)\end{matrix}}{(\Psi\to\neg T(\ulcorner\Psi\urcorner)\vee U_{S_{ITU}}(\ulcorner\Psi\urcorner))\wedge(\neg T(\ulcorner\Psi\urcorner)\vee U_{S_{ITU}}(\ulcorner\Psi\urcorner)\to\Psi)}\ def}{\neg T(\ulcorner\Psi\urcorner)\vee U_{S_{ITU}}(\ulcorner\Psi\urcorner)\to\Psi}\ \wedge E}{\Psi}\ \to E$$

With *SLI*− and *SLE*−inferences, the following theorem and corollary show the inference to $\bot$ and a particular instance of *DNE* from the strengthened liar sentence Ψ.

Theorem 4.1. *If* $S_{ITU}\vdash\Psi\leftrightarrow\neg T(\ulcorner\Psi\urcorner)\vee U_{S_{ITU}}(\ulcorner\Psi\urcorner)$ *and* $S_{ITU}\vdash$ $⅄\to\bot$*, then* $S_{ITU}\vdash\bot$.

Proof. Suppose $S_{ITU}\vdash\Psi\leftrightarrow\neg T(\ulcorner\Psi\urcorner)\vee U_{S_{ITU}}(\ulcorner\Psi\urcorner)$. Then, we can use *SLI*− and *SLE*−inferences. Also, we suppose that $S_{ITU}\vdash ⅄\to\bot$. Then, we have a derivation $\mathfrak{D}_\bot$ of $⅄\to\bot$ in S_{ITU} and an open derivation $\mathfrak{D}_1$ of $\bot$ from $[U_{S_{ITU}}(\ulcorner\Psi\urcorner)]$ and $[\Psi]$.

$$\dfrac{\dfrac{\dfrac{\dfrac{[U_{S_{ITU}}(\ulcorner\Psi\urcorner)]^3}{\nvdash_{S_{ITU}}\Psi\wedge\nvdash_{S_{ITU}}\neg\Psi}\ def}{\nvdash_{S_{ITU}}\Psi}\ \wedge E \qquad [\Psi]^1}{⅄}\ \to E \qquad \begin{matrix}\mathfrak{D}_\bot\\ ⅄\to\bot\end{matrix}}{\bot}\ \to E$$

Having $\mathfrak{D}_1$, the following derivation gives a closed derivation $\mathfrak{D}_2$ of

$\neg\Psi$.

$$\dfrac{\dfrac{[\Psi]^1}{\neg T(\ulcorner\Psi\urcorner)\vee U_{S_{ITU}}(\ulcorner\Psi\urcorner)}SLI \qquad \dfrac{[\neg T(\ulcorner\Psi\urcorner)]^2 \quad \dfrac{[\Psi]^1}{T(\ulcorner\Psi\urcorner)}TI}{\bot}\to E \qquad \begin{matrix}[\Psi]^1,[U_{S_{ITU}}(\ulcorner\Psi\urcorner)]^3\\ \mathfrak{D}_1\\ \bot\end{matrix}}{\dfrac{\bot}{\neg\Psi}\to I_{,1}}\vee E_{,2,3}$$

From the claim 1 of Appendix A, we have a derivation $\mathfrak{D}_{\mathfrak{T}}$ of $\neg T(\ulcorner\Psi\urcorner)\leftrightarrow T(\ulcorner\neg\Psi\urcorner)$ in S_{ITU}. With the derivation $\mathfrak{D}_2$ of $\neg\Psi$, the following process gives a closed derivation $\mathfrak{D}_3$ of Ψ in S_{ITU}.

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_2\\ \neg\Psi\end{matrix}}{T(\ulcorner\neg\Psi\urcorner)}TI \qquad \dfrac{\dfrac{\begin{matrix}\mathfrak{D}_{\mathfrak{T}}\\ \neg T(\ulcorner\Psi\urcorner)\leftrightarrow T(\ulcorner\neg\Psi\urcorner)\end{matrix}}{(\neg T(\ulcorner\Psi\urcorner)\to T(\ulcorner\neg\Psi\urcorner))\wedge(T(\ulcorner\neg\Psi\urcorner)\to\neg T(\ulcorner\Psi\urcorner))}def}{T(\ulcorner\neg\Psi\urcorner)\to\neg T(\ulcorner\Psi\urcorner)}\wedge E}{\neg T(\ulcorner\Psi\urcorner)}\to E}{\dfrac{\neg T(\ulcorner\Psi\urcorner)\vee U_{S_{ITU}}(\ulcorner\Psi\urcorner)}{\Psi}SLE}\vee I$$

Having the derivation $\mathfrak{D}_2$ and $\mathfrak{D}_3$, we have the derivation of $\bot$ in S_{ITU}.

$$\dfrac{\begin{matrix}\mathfrak{D}_2\\ \neg\Psi\end{matrix} \quad \begin{matrix}\mathfrak{D}_3\\ \Psi\end{matrix}}{\bot}\to E$$

Corollary 4.2. *If* $S_{ITU}\vdash\Psi\leftrightarrow\neg T(\ulcorner\Psi\urcorner)\vee U_{S_{ITU}}(\ulcorner\Psi\urcorner)$ *and* $S_{ITU}\vdash \curlywedge\to\bot$*, then* $S_{ITU}\vdash\neg\neg\Psi\to\Psi$.

Proof. From the proof of Theorem 4.1, we have a derivation $\mathfrak{D}_2$ of

Ψ in S_{ITU}.

$$\dfrac{\begin{matrix}\mathfrak{D}_2 \\ \Psi\end{matrix}}{\neg\neg\Psi \rightarrow \Psi} \rightarrow I_{,\varnothing}$$

Theorem 4.1 explicates that even if the truth-valuelessness, $U_{S_{ITU}}$, is expressible in S_{ITU}, there exists an inference to $\perp$. In addition, Corollary 4.2 says that the strengthened liar sentence Ψ has a proof in S_{ITU}. These results do not use $\perp_C$−rule, so one may claim that an intuitionistic system like S_{ITU} does not have any advantage to eliminate the inferences to $\perp$.

5 The Rejection of *DNE* Does Not Block the Inference to $\perp$.

If *POB* and the application of *LEM* have a pivotal role to yield $\perp$ from the liar-type paradoxes, intuitionistic logic has a benefit of preventing the liar-type paradoxes. The strengthened liar paradox in a semantic version presumes that every sentence is either true, false, or truth-valueless. At first glance, the presumption may seem to reject *POB*, but in fact, it is an extended version of *POB*. For an intuitionist, truth-valuelessness is a *gap* between truth and falsity, whereas truth-valuelessness in the semantic strengthened liar is the third value as a *glut* between them. In short, the intuitionist's truth-valuelessness is the prooflessness, yet the truth-valuelessness of the semantic strengthened liar is not.

With respect to the issue of *LEM*, Priest (1983) has attempted to show that *LEM* is unnecessary for the establishment of the derivation of $\perp$ from Berry's paradox. If it were successful, there exists a case that the rejection of *LEM* does not exclude the inference to $\perp$.

Unfortunately, Ross Brady (1984) argues that Priest needs *LEM* to derive $\bot$ from Berry's paradox, and explains how Priest implicitly assumes *LEM*. Priest (2006, pp. 25–27) proposes an alternative argument for the derivation of $\bot$. However, his proof has an application of *DNE* which is provably equivalent to *LEM*. His proof is unsuccessful to show that the inference to $\bot$ exists without *LEM*.

Even if Priest (1983, 2006) fails to establish $\bot$ from Berry's paradox without *LEM*, Corollary 3.2 and Theorem 4.1 show the derivations of $\bot$ without $\bot_C$−rule in the intuitionistic natural deduction systems, S_{IT} and S_{ITU}. Moreover, Corollary 3.3 and 4.2 provide proofs of the instances of *DNE*. If an intuitionistic system has to refuse any instances of *DNE*, both S_{IT} and S_{ITU} are not intuitionistic, or any derivation of an instance of *DNE* is to be restricted. For instance, Corollary 3.3 proves that, for a liar sentence Φ, if $S_{IT} \vdash \Phi \leftrightarrow \neg T(\ulcorner \Phi \urcorner)$, then $S_{IT} \vdash \neg\neg\Phi \rightarrow \Phi$. If the intuitionist has to reject all the instances of *DNE*, (s)he must accept that for every φ in $\mathfrak{L}$, $S_{IT} \nvdash \neg\neg\varphi \rightarrow \varphi$. An application of *modus tollens* to Corollary 3.3 gives $S_{IT} \nvdash \Phi \leftrightarrow \neg T(\ulcorner \Phi \urcorner)$, so the inference to $\bot$ from $\Phi \leftrightarrow \neg T(\ulcorner \Phi \urcorner)$ in S_{IT} is rejected.

The restriction of any instances of *DNE*, however, is too strong because the intuitionist accepts the instance if it has a proof. Furthermore, though every instance of *DNE* is rejected in any intuitionistic systems, the restriction is unable to block every inference to $\bot$. Let us consider any intuitionistic natural deduction system with the following incoherent rules:

$$\dfrac{\begin{array}{c}\mathfrak{D}_1\\ \varphi_i\end{array}}{\varphi_1 \ tonk \ \varphi_2}\,tonkI_{(i=1,2)} \qquad \dfrac{\begin{array}{c}\mathfrak{D}_2\\ \varphi_1 \ tonk \ \varphi_2\end{array}}{\varphi_i}\,tonkE_{(i=1,2)}$$

Then, we easily have a derivation of $\bot$ on the left side below as well as an instance of *DNE* on the right.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi]^1}{\varphi\ tonk\ \bot}\,tonkI_1}{\bot}\,tonkE_2}{\neg\varphi}\to I_{,1}}{\neg\varphi\ tonk\ \bot}\,tonkI_1}{\bot}\,tonkE_2 \qquad \dfrac{\dfrac{\dfrac{[\neg\neg\varphi]^2}{\neg\neg\varphi\ tonk\ \varphi}\,tonkI_1}{\varphi}\,tonkE_2}{\neg\neg\varphi\to\varphi}\to I_{,2}$$

The issue is that the constraint of all instances of *DNE* does not prevent the application of *tonk*-rules. A natural deduction rule can be conceived as a scheme of a particular inference form. $\bot_C$-rule as a scheme is the set of all inference forms deriving a conclusion φ from an open derivation $\begin{matrix}[\neg\varphi]\\ \mathfrak{D}\\ \bot\end{matrix}$. If the restriction of all instances of *DNE* implies the constraint of any inference form of φ from an open derivation $\begin{matrix}[\neg\varphi]\\ \mathfrak{D}\\ \bot\end{matrix}$, it prohibits the use of $\bot_C$-rule. However, it does not mean that any rule which derives an instance of *DNE* should be rejected since all such rules do not share the same inference form. The derivation of $\neg\neg\varphi\to\varphi$ from *tonk*-rules has no open derivation $\begin{matrix}[\neg\varphi]\\ \mathfrak{D}\\ \bot\end{matrix}$. Therefore, the restriction of *DNE* (or of its instances) does not block the inferences to $\bot$.[7]

[7]The requirement of normalizability may eliminate the inference to $\bot$. (Cf. Prawitz (1965) and Tennant (1982, 1995, 2017)). Even if Tennant (1982) embraces the requirement as an intuitionistic requirement, the intuitionistic natural deduction

6 The Intuitionistic 'not true' as the Prooflessness

If our discussions above are correct, the intuitionistic natural deduction systems, S_{IT} and S_{ITU}, have inferences to $\bot$ without the application of $\bot_C$ and the restriction of *DNE* does not block the inference. However, it does not mean that any intuitionistic formal systems have no advantage to rule out the inference. We shall consider an intuitionistic natural deduction system S_I with the meta-level absurdity $\curlywedge$ and see that it blocks the inferences to $\bot$ if $\bot \to \curlywedge$ is true.

As noted in Section 4 and 5, 'φ is not true' is intuitionistically conceived in two ways: $\neg\varphi$ has a proof, i.e. φ has a disproof, and φ has no proof. For a given system S, '$\neg\varphi$ has a proof' is expressed in S by '$S \vdash \neg\varphi$' and 'φ has no proof' is by '$S \nvdash \varphi$.' To express the prooflessness in an intuitionistic system, we presume that '$\nvdash_{S_{I'}}$' is expressible in the language of an intuitionistic natural deduction system $S_{I'}$. We read 'φ is not true in $S_{I'}$' as $\nvdash_{S_{I'}} \varphi$, so to speak, $\varphi \to \curlywedge$.

We define a formula Φ as $\nvdash_{S_{I'}} \Phi$. We say that φ is a *liar* sentence in $S_{I'}$ if φ satisfies the relation $\varphi \leftrightarrow \nvdash_{S_{I'}} \varphi$. Then, we have the following theorems.

Theorem 6.1. *If* $S_{I'} \vdash \Phi \leftrightarrow \nvdash_{S_{I'}} \Phi$ *and* $S_{I'} \vdash \bot \to \curlywedge$, *then* $S_{I'} \vdash \nvdash_{S_{I'}} \Phi \wedge \nvdash_{S_{I'}} \neg\Phi$.

Proof. Suppose that $S_{I'}$ has a closed derivation of $\mathfrak{D}_{\nvdash}$ of $\Phi \leftrightarrow \nvdash_{S_{I'}} \Phi$

system is not a unique system that satisfies the requirement. Prawitz (1965) has proven the *Normal Form Theorem* for a weak classical logic. The proof of the same theorem for a full classical logic was suggested by Stålmarck (1991). Normalizability is the property of the general proof-theory, but not only of intuitionistic logic. In this paper, we focus on intuitionistic approaches to the liar-type paradoxes in the intuitionistic natural deduction system. We set aside the issue of normalizability.

and $\mathfrak{D}_{\Yup}$ of $\bot \to \Yup$. Then, we have a closed derivation $\mathfrak{D}_1$ of $\nvdash_{S_{I'}} \Phi$.

$$\dfrac{\dfrac{[\Phi]^1 \qquad \dfrac{[\Phi]^1 \qquad \dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_{\nvdash}\\ \Phi \leftrightarrow \nvdash_{S_{I'}} \Phi\end{array}}{(\Phi \to \nvdash_{S_{I'}} \Phi) \wedge (\nvdash_{S_{I'}} \Phi \to \Phi)}\,def}{\Phi \to \nvdash_{S_{I'}} \Phi}\,\wedge E}{\nvdash_{S_{I'}} \Phi}\,\to E}{\Yup}\,\to E}{\nvdash_{S_{I'}} \Phi}\,\to I_{,1}$$

Having $\mathfrak{D}_1$, we have a closed derivation $\mathfrak{D}_2$ of $\nvdash_{S_{I'}} \neg\Phi$.

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_{\Yup}\\ \bot \to \Yup\end{array} \qquad \dfrac{[\neg\Phi]^1 \qquad \dfrac{\begin{array}{c}\mathfrak{D}_1\\ \nvdash_{S_{I'}} \Phi\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_{\nvdash}\\ \Phi \leftrightarrow \nvdash_{S_{I'}} \Phi\end{array}}{(\Phi \to \nvdash_{S_{I'}} \Phi) \wedge (\nvdash_{S_{I'}} \Phi \to \Phi)}\,def}{\nvdash_{S_{I'}} \Phi \to \Phi}\,\wedge E}{\Phi}\,\to E}{\bot}\,\to E}{\Yup}\,\to E}{\nvdash_{S_{I'}} \neg\Phi}\,\to I_{,1}$$

Hence, from $\mathfrak{D}_1$ of $\nvdash_{S_{I'}} \Phi$ and $\mathfrak{D}_2$ of $\nvdash_{S_{I'}} \neg\Phi$, we have $\nvdash_{S_{I'}} \Phi \wedge \nvdash_{S_{I'}} \neg\Phi$. □

In addition, the inference to $\bot$ is derivable in $S_{I'}$ if $\Yup \to \bot$.

Theorem 6.2. *If $S_{I'} \vdash \Phi \leftrightarrow \nvdash_{S_{I'}} \Phi$ and $S_{I'} \vdash \Yup \to \bot$, then $S_{I'} \vdash \bot$.*

Proof. From the proof of Theorem 6.1, we have a derivation $\mathfrak{D}_1$ of $\nvdash_{S_{I'}} \Phi$. Suppose that $S_{I'} \vdash \Yup \to \bot$. Then, there exists a closed derivation $\mathfrak{D}_{\perp}$ of $⅄ \to \perp$. We have a derivation of $\perp$ as follows:

$$
\dfrac{\begin{array}{c}\mathfrak{D}_1\\ ⅄ \to \perp\end{array} \qquad \dfrac{\begin{array}{c}\mathfrak{D}_1\\ \nvdash_{S_{I'}} \Phi\end{array} \qquad \dfrac{\begin{array}{c}\mathfrak{D}_1\\ \nvdash_{S_{I'}} \Phi\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_{\nvdash}\\ \Phi \leftrightarrow \nvdash_{S_{I'}} \Phi\end{array}}{(\Phi \to \nvdash_{S_{I'}} \Phi) \wedge (\nvdash_{S_{I'}} \Phi \to \Phi)}\, def}{\nvdash_{S_{I'}} \Phi \to \Phi}\, \wedge E}{\Phi}\to E}{⅄}\to E}{\perp}\to E
$$

$\square$

Theorem 6.2 has the same result of Corollary 3.2, even if they are proved in different systems, $S_{I'}$ and S_{IT}. With the supposition that $S_{I'} \vdash \perp \to ⅄$, Theorem 6.1 says that neither Φ nor $\neg\Phi$ is derivable in $S_{I'}$. Theorem 6.1 and 6.2 are relative to the suppositions of $S_{I'} \vdash ⅄ \to \perp$ and $S_{I'} \vdash \perp \to ⅄$. A short investigation shows that $S_{I'} \vdash ⅄ \to \perp$ implies that $S_{I'}$ is complete, and $S_{I'} \vdash \perp \to ⅄$ leads to the consistency of $S_{I'}$. The derivation below says that, for any φ in $\mathfrak{L}$, if $⅄ \to \perp$, then $\nvdash_{S_{I'}} \varphi \to \neg\varphi$. Also, $\nvdash_{S_{I'}} \perp$ if $\perp \to ⅄$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\nvdash_{S_{I'}} \varphi]^1 \quad [\varphi]^2}{⅄}\to E \qquad [⅄ \to \perp]^3}{\perp}\to E}{\neg\varphi}\to I_{,2}}{\nvdash_{S_{I'}} \varphi \to \neg\varphi}\to I_{,1}}{(⅄ \to \perp) \to (\nvdash_{S_{I'}} \varphi \to \neg\varphi)}\to I_{,3}
\qquad
\dfrac{\dfrac{\dfrac{[\perp]^4 \quad [\perp \to ⅄]^5}{⅄}\to E}{\nvdash_{S_{I'}} \perp}\to I_{,4}}{(\perp \to ⅄) \to \nvdash_{S_{I'}} \perp}\to I_{,5}
$$

We call the left derivation $\mathfrak{D}_{\mathfrak{M}}$ and the right $\mathfrak{D}_{\mathfrak{N}}$. Say, for any system S, S is consistent if $S \nvdash \perp$; otherwise, inconsistent. $\mathfrak{D}_{\mathfrak{M}}$ explains that $S_{I'} \vdash ⅄ \to \perp$ implies that $S_{I'}$ is complete and $\mathfrak{D}_{\mathfrak{N}}$ says that $S_{I'} \vdash \perp \to$

$\Yup$ implies that $S_{I'}$ is consistent. Theorem 6.1 and 6.2 show the relation between completeness and consistency. When the language $\mathfrak{L}$ of $S_{I'}$ has a liar sentence, $S_{I'}$ is not both complete and consistent.[8]

Without the assumption that $\Yup \to \bot$ in Theorem 6.2, there is no inference to $\bot$ from $\Phi \leftrightarrow \nvdash_{S_{I'}} \Phi$. Likewise, Theorem 4.1 establishes that an inference to $\bot$ occurs if $S_{ITU} \vdash \Psi \leftrightarrow \neg T(\ulcorner \Psi \urcorner) \vee U_{S_{ITU}}(\ulcorner \Psi \urcorner)$ and $S_{ITU} \vdash \Yup \to \bot$. The inference to $\bot$ exists only if the system in question is complete. Since there is no reason for the intuitionist that a correct intuitionistic formal system must be complete, an intuitionistic natural deduction system $S_{I'}$ has the advantage to block the inference to $\bot$.

A plausible objection can be suggested by the fact that if $S_{I'} \vdash \Phi \leftrightarrow \nvdash_{S_{I'}} \Phi$, then $S_{I'} \vdash \Yup$. From the proof of Theorem 6.1, we have a derivation $\mathfrak{D}_1$ of $\nvdash_{S_{I'}} \Phi$. Then, the following process proves the fact.

$$\dfrac{\begin{matrix}\mathfrak{D}_1\\ \nvdash_{S_{I'}}\Phi\end{matrix} \qquad \dfrac{\begin{matrix}\mathfrak{D}_1\\ \nvdash_{S_{I'}}\Phi\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\mathfrak{D}_{\nvdash}\\ \Phi\leftrightarrow\nvdash_{S_{I'}}\Phi\end{matrix}}{(\Phi\to\nvdash_{S_{I'}}\Phi)\wedge(\nvdash_{S_{I'}}\Phi\to\Phi)}\,def}{\nvdash_{S_{I'}}\Phi\to\Phi}\,\wedge E}{\Phi}\,\to E}{\Yup}\,\to E$$

Even if there is no inference to $\bot$ from $\Phi \leftrightarrow \nvdash_{S_{I'}} \Phi$, without the assumption of $S_{I'} \vdash \Yup \to \bot$, the fact shows that there exists the inference to $\Yup$. Be that as it may, the proof is dependent on the derivation

[8] Even in an axiomatic system, we can have the similar result. For example, Choi (2017) argues that the lesson of Gödel's proof is that any sufficiently strong and intuitively correct arithmetic cannot both be complete and consistent. That is, the threats of inconsistency generated by the liar-type sentences or Gödel sentence would be occured only in complete formal systems but not in others.

$\mathfrak{D}_{\nvdash}$ of $\Phi \leftrightarrow \nvdash_{S_{I'}} \Phi$. If we do not suppose that $S_{I'}$ has the derivation $\mathfrak{D}_{\nvdash}$, we have the following result.

Theorem 6.3. $S_{I'} \vdash \nvdash_{S_{I'}} (\Phi \leftrightarrow \nvdash_{S_{I'}} \Phi)$.

Proof. There exists an open derivation
$$\begin{array}{c} [\Phi \leftrightarrow \nvdash_{S_{I'}} \Phi] \\ \mathfrak{D}_1 \\ \nvdash_{S_{I'}} \Phi \end{array}.$$

$$\dfrac{\dfrac{[\Phi]^2 \quad \dfrac{[\Phi]^2 \quad \dfrac{\dfrac{[\Phi \leftrightarrow \nvdash_{S_{I'}} \Phi]^1}{(\Phi \to \nvdash_{S_{I'}} \Phi) \wedge (\nvdash_{S_{I'}} \Phi \to \Phi)}\,def}{\Phi \to \nvdash_{S_{I'}} \Phi}\,\wedge E}{\nvdash_{S_{I'}} \Phi}\,\to E}{\curlywedge}\,\to E}{\nvdash_{S_{I'}} \Phi}\,\to I_{,2}$$

Then, we have a proof of $\nvdash_{S_{I'}} (\Phi \leftrightarrow \nvdash_{S_{I'}} \Phi)$ with the open derivation $\mathfrak{D}_1$.

$$\dfrac{\dfrac{\begin{array}{c}[\Phi \leftrightarrow \nvdash_{S_{I'}} \Phi]^1 \\ \mathfrak{D}_1 \\ \nvdash_{S_{I'}} \Phi\end{array} \quad \dfrac{\begin{array}{c}[\Phi \leftrightarrow \nvdash_{S_{I'}} \Phi]^1 \\ \mathfrak{D}_1 \\ \nvdash_{S_{I'}} \Phi\end{array} \quad \dfrac{\dfrac{[\Phi \leftrightarrow \nvdash_{S_{I'}} \Phi]^1}{(\Phi \to \nvdash_{S_{I'}} \Phi) \wedge (\nvdash_{S_{I'}} \Phi \to \Phi)}\,def}{\nvdash_{S_{I'}} \Phi \to \Phi}\,\wedge E}{\Phi}\,\to E}{\curlywedge}\,\to E}{\nvdash_{S_{I'}} (\Phi \leftrightarrow \nvdash_{S_{I'}} \Phi)}\,\to I_{,1}$$

With the derivation $\mathfrak{D}_{\nvdash}$ of $\Phi \leftrightarrow \nvdash_{S_{I'}} \Phi$, the liar sentence Φ has a privilege to guarantee that a prooflessness implies an existence of a proof. The intuitionist can claim that any sentence which lacks a proof-condition is unable to have a consequence, so the relation from

$\nvdash_{S_{I'}} \Phi$ to Φ is unacceptable. This is not a mere proposal to reject the assumption that the liar-type sentences, such as $\Phi \leftrightarrow \nvdash_{S_{I'}} \Phi$, can be defined in the language of $S_{I'}$. Even though $\Phi \leftrightarrow \nvdash_{S_{I'}} \Phi$ can be defined in the language, it is proved by Theorem 6.3 that $\Phi \leftrightarrow \nvdash_{S_{I'}} \Phi$ is not derivable in $S_{I'}$. Therefore, with the expression of the prooflessness, $\nvdash_{S_{I'}}$, and the liar sentence in terms of it, there is no inference to $\bot$ in $S_{I'}$ unless $S_{I'}$ is complete.

7 Rules for the *Double Meta-Level Negation Elimination*

In this paper, we do not claim that a genuine intuitionistic negation for a given system S must be $\nvdash_S$. Rather, we focus on the case that a negation, $\neg$, can express neither the prooflessness nor the intuitionistic meanings of 'not true.' A classical interpretation of 'φ is not true' is the same as 'φ is false' (or '$\neg\varphi$ is true'). A classical formal system does not need to express the prooflessness, so the meta-level negation, $\nvdash_S$, and the meta-level absurdity, $\curlywedge$, are not necessary.

When the system $S_{I'}$ has the meta-level negation and its language has a liar sentence Φ which satisfies the relation $\Phi \leftrightarrow \nvdash_{S_{I'}} \Phi$, $S_{I'}$ has no inference to $\bot$ unless $S_{I'}$ is complete. As we have seen in Section 6, $S_{I'}$ prevents the inference to $\bot$. However, it does not change the fact that the restriction of *DNE* does not block the inference to $\bot$. Tennant (2017, p.284), considering the similar proof of Corollary 3.2, claims, '[*LEM*] need not play any role in the derivation of the explicit contradiction [$\Phi \wedge \neg\Phi$] from [$\Phi \leftrightarrow \neg\Phi$].' Also, the proof of Theorem 6.2 makes no use of $\bot_C$-rule and the assumption of $\curlywedge \rightarrow \bot$ does not presume the use of it. *DNE* and *LEM* in the standard intuitionistic natural deduction system may not concern about the meta-level negation (or prooflessness). Instead of *DNE* and *LEM*, we may consider the *Double Meta-Level Negation Elimination* (*DMNE*) which implicitly assumes that a given system is complete.

With their interpretations of truth, intuitionists, such as Michael Dummett (1973) and Tennant (1997), have given philosophical arguments against *POB*. For a given system S, *POB* guarantees that S is complete since it says that every sentence φ in $\mathfrak{L}$ has a proof or a disproof. *POB* forces us to claim that $S \vdash \varphi$ or $S \vdash \neg\varphi$. Hence, $\nvdash_S \varphi$ implies $\neg\varphi$ and $\nvdash_S \neg\varphi$ implies φ. Under the assumption of *POB*, we may accept the following rules for $\curlywedge$ with respect to a given system S.

$$\dfrac{\begin{matrix}[\varphi]^1 \\ \mathfrak{D}_1 \\ \curlywedge\end{matrix}}{\neg\varphi}\,\curlywedge_{C_1,1} \qquad\qquad \dfrac{\begin{matrix}[\nvdash_S \varphi]^2 \\ \mathfrak{D}_2 \\ \curlywedge\end{matrix}}{\varphi}\,\curlywedge_{C_2,2}$$

We add the rules above to $S_{I'}$ and have $S_{I'D}$. $S_{I'D}$ has the same result of Theorem 6.2 without the assumption of $\curlywedge \to \bot$.

Theorem 7.1. *If $S_{I'D} \vdash \Phi \leftrightarrow \nvdash_{S_{I'D}} \Phi$, then $S_{I'D} \vdash \bot$.*

Proof. Suppose $S_{I'D}$ has a closed derivation of $\mathfrak{D}_{\nvdash}$ of $\Phi \leftrightarrow \nvdash_{S_{I'D}} \Phi$. We have a derivation $\mathfrak{D}_1$ of Φ as follows:

$$\dfrac{\dfrac{[\nvdash_{S_{I'D}} \Phi]^1 \qquad \dfrac{[\nvdash_{S_{I'D}} \Phi]^1 \qquad \dfrac{\dfrac{\begin{matrix}\mathfrak{D}_{\nvdash} \\ \Phi \leftrightarrow \nvdash_{S_{I'D}} \Phi\end{matrix}}{(\Phi \to \nvdash_{S_{I'D}} \Phi) \wedge (\nvdash_{S_{I'D}} \Phi \to \Phi)}\,def}{\nvdash_{S_{I'D}} \Phi \to \Phi}\,\wedge E}{\Phi}\,\to E}{\curlywedge}\,\to E}{\Phi}\,\curlywedge_{C_2,1}$$

Also, we have a derivation $\mathfrak{D}_2$ of $\neg\Phi$ below.

$$\cfrac{\cfrac{[\Phi]^2 \qquad \cfrac{[\Phi]^2 \qquad \cfrac{\cfrac{\begin{array}{c}\mathfrak{D}_{\nvdash}\\ \Phi \leftrightarrow \nvdash_{S_{I'D}} \Phi\end{array}}{(\Phi \to \nvdash_{S_{I'D}} \Phi) \wedge (\nvdash_{S_{I'D}} \Phi \to \Phi)}\,def}{\Phi \to \nvdash_{S_{I'D}} \Phi}\,\wedge E}{\nvdash_{S_{I'D}} \Phi}\,\to E}{\curlywedge}\,\to E}{\neg\Phi}\,\curlywedge_{C_1,2}$$

Having $\mathfrak{D}_1$ and $\mathfrak{D}_2$, we have a derivation of $\bot$.

$$\cfrac{\begin{array}{cc}\mathfrak{D}_1 & \mathfrak{D}_2\\ \Phi & \neg\Phi\end{array}}{\bot}\,\to E$$

Similarly, if S_{ITU} has $\curlywedge_C-$rules, Theorem 4.1 is established without the assumption of $\curlywedge \to \bot$. (See Appendix C.) $\curlywedge_{C_2}-$rule has a similar form of $\bot_C-$rule. We may regard $\curlywedge_{C_1}-$ and $\curlywedge_{C_2}-$rules as the rules for *DMNE*. $\curlywedge_C-$rules assume that a given system is complete and such assumption is justified by *POB*. In this perspective, the rejection of *POB* and *DMNE* can exclude the inference to $\bot$ in $S_{I'D}$ (and in S_{ITUD} of Appendix C.)

Consequently, when we regard 'not true' as prooflessness, our discussions from Section 6 and 7 explicate that $S_{I'}$ has no threat to derive $\bot$ unless $S_{I'}$ is complete. Therefore, *DMNE* has a significant role in deriving $\bot$ and the restriction of *DMNE* blocks the inference to $\bot$.

8 Conclusion

As is often said that, in a purely technical consideration, an intuitionistic formal system has no advantage to rule out the inference to $\bot$ from the liar-type paradoxes. We presume that a standard intuitionistic natural deduction system is a primary candidate formal system for intuitionistic logic. By the arguments from Section 3 to 5, we have seen that intuitionistic systems, S_{IT} and S_{ITU}, are not immune to the liar-type paradoxes. Also, since the proofs of Corollary 3.2 and Theorem 4.1 do not use $\bot_C-$rule, the restriction of $\bot_C-$rule (or *DNE*) does not block the inference to $\bot$ from the paradoxes.

In Section 6 and 7, we have argued that it is not a problem of intuitionistic strategies to the liar-type paradoxes, but the lack of expressive power of S_{IT} and S_{ITU} in that S_{IT} and S_{ITU} are unable to express an intuitionistic 'not true.' In the reinforced system $S_{I'}$ having $\nvdash_{S_{I'}}$ and $\curlywedge$, there is no inference to $\bot$ if $S_{I'}$ is not complete. Furthermore, as long as 'not true' is interpreted as prooflessness, we may claim that no inference to $\bot$ occur if *DMNE* is restricted in $S_{I'}$. Therefore, the intuitionistic system $S_{I'}$ can defend the challenge that intuitionistic logic has no advantage to block the liar-type paradoxes.

It is a controversial point whether an intuitionistic system should have a meta-level negation and a meta-level absurdity constant. As the claim 1 of Appendix A shows, S_I and S_{IT} cannot distinguish between 'φ has a disproof' and 'φ has no proof.' It may be implicitly assumed that S_I and S_{IT} are complete. If it is an intuitionist's problem that a negation, $\neg$, in an intuitionistic system cannot express the prooflessness, an application of the meta-level negation and the meta-level absurdity to the system may be a right answer.

In sum, some extensions of the standard intuitionistic natural deduction system are vulnerable to inconsistency from the liar-type paradoxes, such as S_{IT} and S_{ITU}. On the other hand, some, such as $S_{I'}$, are not. Therefore, in a purely technical perspective, not all the intuitionistic natural deduction system fail to rule out the inference to $\bot$ from the liar-type paradoxes.

Appendix A: Some Facts in S_{IT}

In this appendix, we give a proof of the fact that if $S_{IT} \vdash \Phi \leftrightarrow \neg T(\ulcorner\Phi\urcorner)$, then $S_{IT} \vdash \neg\Phi \leftrightarrow T(\ulcorner\Phi\urcorner)$. The proof consists of three claims.

Claim 1. $S_{IT} \vdash \neg T(\ulcorner\Phi\urcorner) \leftrightarrow T(\ulcorner\neg\Phi\urcorner)$.

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg T(\ulcorner\Phi\urcorner)]^1 \quad \dfrac{[\Phi]^2}{T(\ulcorner\Phi\urcorner)}\,TI}{\bot}\to E}{\neg\Phi}\to I_{,2}}{T(\ulcorner\neg\Phi\urcorner)}\,TI}{\neg T(\ulcorner\Phi\urcorner) \to T(\ulcorner\neg\Phi\urcorner)}\to I_{,1} \quad \dfrac{\dfrac{\dfrac{\dfrac{[T(\ulcorner\Phi\urcorner)]^3}{\Phi}\,TE \quad \dfrac{[T(\ulcorner\neg\Phi\urcorner)]^4}{\neg\Phi}\,TE}{\bot}\to E}{\neg T(\ulcorner\Phi\urcorner)}\to I_{,3}}{T(\ulcorner\neg\Phi\urcorner) \to \neg T(\ulcorner\Phi\urcorner)}\to I_{,4}}{(\neg T(\ulcorner\Phi\urcorner) \to T(\ulcorner\neg\Phi\urcorner)) \wedge (T(\ulcorner\neg\Phi\urcorner) \to \neg T(\ulcorner\Phi\urcorner))}\wedge I}{\neg T(\ulcorner\Phi\urcorner) \leftrightarrow T(\ulcorner\neg\Phi\urcorner)}\,def$$

Claim 2. If $S_{IT} \vdash \Phi \leftrightarrow \neg T(\ulcorner\Phi\urcorner)$, then $S_{IT} \vdash T(\ulcorner\Phi\urcorner) \to \neg\Phi$.

Proof. Suppose that $S_{IT} \vdash \Phi \leftrightarrow \neg T(\ulcorner\Phi\urcorner)$. Then, we can use LI_1-

and LE_1-inferences.

$$\dfrac{\dfrac{\dfrac{[T(\ulcorner\Phi\urcorner)]^1 \quad \dfrac{[\Phi]^2}{\neg T(\ulcorner\Phi\urcorner)}\,LI_1}{\bot}\,\to E}{\neg\Phi}\,\to I_{,2}}{T(\ulcorner\Phi\urcorner)\to\neg\Phi}\,\to I_{,1}$$

Claim 3. If $S_{IT}\vdash\Phi\leftrightarrow\neg T(\ulcorner\Phi\urcorner)$, then $S_{IT}\vdash\neg\Phi\to T(\ulcorner\Phi\urcorner)$.

Proof. With the claim 1, we suppose that $S_{IT}\vdash\Phi\leftrightarrow\neg T(\ulcorner\Phi\urcorner)$ and use LI_1- and LE_1-inferences.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg\Phi]^1}{T(\ulcorner\neg\Phi\urcorner)}\,TI \quad \dfrac{\dfrac{\dfrac{}{\neg T(\ulcorner\Phi\urcorner)\leftrightarrow T(\ulcorner\neg\Phi\urcorner)}\,\text{claim 1}}{(\neg T(\ulcorner\Phi\urcorner)\to T(\ulcorner\neg\Phi\urcorner))\wedge(T(\ulcorner\neg\Phi\urcorner)\to\neg T(\ulcorner\Phi\urcorner))}\,def}{T(\ulcorner\neg\Phi\urcorner)\to\neg T(\ulcorner\Phi\urcorner))}\,\wedge E}{\neg T(\ulcorner\Phi\urcorner)}\,\to E}{\Phi}\,LE_1}{T(\ulcorner\Phi\urcorner)}\,TI}{\neg\Phi\to T(\ulcorner\Phi\urcorner)}\,\to I_{,1}$$

Therefore, by the three claims, if $S_{IT}\vdash\Phi\leftrightarrow\neg T(\ulcorner\Phi\urcorner)$, then $S_{IT}\vdash\neg\Phi\leftrightarrow T(\ulcorner\Phi\urcorner)$.

Appendix B: The Proof of $\bot$ by *LEM*

From Theorem 3.1, we have the same result of Corollary 3.2 with the application of *LEM*. We give a rule for *LEM* as follows.

$$\dfrac{\begin{matrix}[\varphi]^1 & [\neg\varphi]^2 \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ \psi & \psi\end{matrix}}{\psi}\, LEM_{,1,2}$$

Then, we have a derivation of $\bot$.

$$\dfrac{\dfrac{[\Phi]^1 \quad \dfrac{[\Phi]^1 \quad \dfrac{\dfrac{\dfrac{}{\Phi\leftrightarrow\neg\Phi}\,\text{theorem 3.1}}{(\Phi\to\neg\Phi)\wedge(\neg\Phi\to\Phi)}\,def}{\Phi\to\neg\Phi}\,\wedge E}{\neg\Phi}\,\to E}{\bot}\,\to E \qquad \dfrac{[\neg\Phi]^2 \quad \dfrac{[\neg\Phi]^2 \quad \dfrac{\dfrac{\dfrac{}{\Phi\leftrightarrow\neg\Phi}\,\text{theorem 3.1}}{(\Phi\to\neg\Phi)\wedge(\neg\Phi\to\Phi)}\,def}{\neg\Phi\to\Phi}\,\wedge E}{\Phi}\,\to E}{\bot}\,\to E}{\bot}\, LEM_{,1,2}$$

As we have seen in Corollary 3.2, however, the same result can be given without the application of *LEM* and $\bot_C-$rule.

Appendix C: The Strengthened Liar Paradox and $\curlywedge_C-$Rules

We have a system S_{ITUD} by adding $\curlywedge_C-$rules to S_{ITU}. Then, we have the following theorem.

Theorem 8.1. *If* $S_{ITUD} \vdash \Psi \leftrightarrow \neg T(\ulcorner\Psi\urcorner) \vee U_{S_{ITUD}}(\ulcorner\Psi\urcorner)$*, then* $S_{ITUD} \vdash \bot$.

Proof. Suppose $S_{ITUD} \vdash \Psi \leftrightarrow \neg T(\ulcorner\Psi\urcorner) \vee U_{S_{ITUD}}(\ulcorner\Psi\urcorner)$. Then, we can use *SLI*$-$ and *SLE*$-$inferences. We have an open derivation $\mathfrak{D}_1$

of $\bot$ from $[U_{S_{ITUD}}(\ulcorner\Psi\urcorner)]$ and $[\Psi]$ below.

$$\dfrac{[U_{S_{ITU}}(\ulcorner\Psi\urcorner)]^4 \qquad \dfrac{\dfrac{\dfrac{\dfrac{[U_{S_{ITUD}}(\ulcorner\Psi\urcorner)]^3}{\nvdash_{S_{ITUD}}\Psi\wedge\nvdash_{S_{ITUD}}\neg\Psi}\,def}{\nvdash_{S_{ITUD}}\Psi}\,\wedge E \qquad [\Psi]^1}{\curlywedge}\,\to E}{\neg U_{S_{ITUD}}(\ulcorner\Psi\urcorner)}\,\curlywedge C_{1,3}}{\bot}\,\to E$$

With $\mathfrak{D}_1$, the following derivation gives a derivation $\mathfrak{D}_2$ of $\neg\Psi$.

$$\dfrac{\dfrac{\dfrac{[\Psi]^1}{\neg T(\ulcorner\Psi\urcorner)\vee U_{S_{ITUD}}(\ulcorner\Psi\urcorner)}\,SLI \qquad \dfrac{[\neg T(\ulcorner\Psi\urcorner)]^2 \quad \dfrac{[\Psi]^1}{T(\ulcorner\Psi\urcorner)}\,TI}{\bot}\,\to E \qquad \begin{array}{c}[\Psi]^1,[U_{S_{ITUD}}(\Psi)]^4\\ \mathfrak{D}_1\\ \bot\end{array}}{\bot}\,\vee E_{,2,4}}{\neg\Psi}\,\to I_{,1}$$

From the claim 1 of Appendix A, we have a derivation $\mathfrak{D}_{\mathfrak{T}}$ of $\neg T(\ulcorner\Psi\urcorner)\leftrightarrow T(\ulcorner\neg\Psi\urcorner)$ in S_{ITUD}. With the derivation $\mathfrak{D}_2$ of $\neg\Psi$, the following process gives a derivation $\mathfrak{D}_3$ of Ψ in S_{ITUD}.

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_2\\ \neg\Psi\end{array}}{T(\ulcorner\neg\Psi\urcorner)}\,TI \qquad \dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_{\mathfrak{T}}\\ \neg T(\ulcorner\Psi\urcorner)\leftrightarrow T(\ulcorner\neg\Psi\urcorner)\end{array}}{(\neg T(\ulcorner\Psi\urcorner)\to T(\ulcorner\neg\Psi\urcorner))\wedge(T(\ulcorner\neg\Psi\urcorner)\to\neg T(\ulcorner\Psi\urcorner))}\,def}{T(\ulcorner\neg\Psi\urcorner)\to\neg T(\ulcorner\Psi\urcorner)}\,\wedge E}{\neg T(\ulcorner\Psi\urcorner)}\,\to E}{\neg T(\ulcorner\Psi\urcorner)\vee U_{S_{ITUD}}(\ulcorner\Psi\urcorner)}\,\vee I}{\Psi}\,SLE$$

Having the derivation $\mathfrak{D}_2$ and $\mathfrak{D}_3$, we have the derivation of $\bot$ in

S_{ITUD}.

$$\dfrac{\begin{matrix}\mathfrak{D}_2 & \mathfrak{D}_3 \\ \neg\Psi & \Psi\end{matrix}}{\bot} \rightarrow E$$

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거짓말쟁이 유형 역설과 직관주의 자연연역체계

최 승 락

순수하게 형식적인 견지에서 직관주의 논리는 거짓말쟁이 유형의 역설을 다루는데 어떠한 이점도 없다고 여겨진다. 이 글에서 우리는 표준 직관주의 자연연역체계가 거짓말쟁이 유형의 역설에 취약함을 논할 것이다. 다시 말해, 거짓말쟁이 유형의 문장을 수용함이 모순(⊥)을 도출하는 추론을 야기한다는 것이다. 이러한 결과는 이중부정 제거규칙(*DNE*)에 대한 제약이 ⊥을 도출하는 추론을 막지 못한다는 것을 보여준다. 하지만 이는 거짓말쟁이 유형의 역설에 대한 직관주의적 접근법이 잘못된 것이 아니라 표준 자연연역 체계의 표현력이 부족한 문제라고 할 수 있다.

우리는 주어진 체계 S에 대한 메타-레벨 부정 연산자 $\nvdash_S$와 메타-레벨 모순 연산자 ⅄를 직관주의 체계에 도입할 것이다. 그리고 체계의 완전성에 대한 가정 없이는 이 체계에서 ⊥에 대한 추론을 얻을 수 없음을 보일 것이다. 또한 우리는 이중 메타-레벨 부정 제거규칙(*DMNE*)을 고려할 것이다. 이 규칙은 체계의 완전성을 암묵적으로 가정하며 *DMNE*에 대한 제약은 ⊥의 추론을 막을 수 있을 것이다.

주요어: 거짓말쟁이 역설, 강화된 거짓말쟁이 역설, 보복 거짓말, 자연연역, 이중부정 제거규칙, 직관주의 논리.