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Algebraic Kripke-Style Semantics for Weakly Associative Fuzzy Logics*

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【Abstract】This paper deals with Kripke-style semantics, which will be called *algebraic* Kripke-style semantics, for weakly associative fuzzy logics. First, we recall algebraic semantics for weakly associative logics. W next introduce algebraic Kripke-style semantics, and also connect them with algebraic semantics.

【Key Words】(Algebraic) Kripke-style semantics, weakening-free fuzzy logic, weak associativity, algebraic semantics, substructural logic.

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1. Introduction

In this paper, we investigate Kripke-style semantics, called *algebraic* Kripke-style semantics, for weakly associative fuzzy logics. These logics are *substrucutural logics*, which are lacking structural rules such as weakening and contraction, with weak forms of associativity in place of associativity itself.

For this, let us first recall some relationships between substructural fuzzy logics and (algebraic) Kripke-style semantics. After Kripke first introduced the so-called *Kripke semantics* for modal and intuitionistic logics in Kripke (1963; 1965a; 1965b) using binary accessibility relations, many semantics generalizing them, the so-called *Kripke-style semantics*, have been provided for many-valued logics. As Yang (2014a; 2016a) mentioned, there are at least two trends in generalization for many-valued logics. One is to provide model structures with binary relations, but without operations. The other is to provide model structures with both operations and binary relations.[1)]

This work is related with the second trend. First, recall his statements on the second trend introduced in Yang (2016a).

> The other trend is to provide model structures with both operations and binary relations. Various types of these semantics have been provided for infinite-valued or fuzzy logics (see Diaconescu & Georgescu

[1)] In Yang (2014b), he introduced three trends. But we follow his consideration in Yang (2014a). Because the other trend is to provide model structures with at least ternnary relations. Here we are only interested in Kripke-style semantics based on binary relations.

(2007), Montagna & Ono (2002), Montagna & Sacchetti (2003), Yang (2012b)). In particular, after semantics for the infinite-valued Łukasiewicz logic **Ł** was introduced by Urquhart (1986), many Kripke-style semantics were recently provided for fuzzy logics based on t-norms (so called t-norm-based logics) by Montagna-Ono (2002), Montagna-Sacchetti (2003; 2004), and Diaconescu-Georgescu (2007). These logics all have *both operational and binary relational* semantics. Thus, semantics with this trend are said to be *operational and binary relational* Kripke-style semantics(Yang (2016a), p. 297).

One important and interesting kind of this trend is to study so called *algebraic* Kripke-style semantics, which are Kripke-style semantics being equivalent to algebraic semantics in that completeness is provided by this equivalence.

Note that some of the authors introduced in the above citation provided algebraic Kripke-style semantics for t-norm[2] based logics after algebraic semantics for those logics were first introduced (Montagna-Ono (2002), Montagna-Sacchetti (2003; 2004)). Yang has generalized this idea to uninorm[3] based logics. That is, he has introduced algebraic Kripke-style semantics for uninorm based logics in Yang (2012; 2014a; 2016a).

T-norms and uninorms require associativity. Recently, some logicians have introduced a non-associative generalization of logics based on t-norms and uninorms (see Cintula & Noguera (2011), Cintula et al (2013, 2015), Horĉík (2011), Yang (2015a; 2015b; 2016c; 2017a; 2017b). In particular, Yang (2016b; 2017c) has

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2) T−norms are commutative, associative, monotonic, binary functions on the unit interval [0, 1] with identity elements 1.

3) Uninorms are t-norms having their identity lying somewhere in [0, 1].

introduced weakly associative fuzzy logics as fuzzy logics with weak forms of associativity in place of associativity itself. However, for these logics, he has considered only algebraic semantics. This gives rise to the following natural question:

- Do algebraically complete non-associative fuzzy logics also have algebraic Kripke-style semantics?

This paper gives a positive answer for some systems as a starting point of this work. More precisely, we provide algebraic Kripke-style semantics for the weakly associative fuzzy logics introduced in Yang (2016b). For this, first, in Section 2 we recall the wta-monoidal uninorm logic $\mathbf{WA_tMUL}$ and its axiomatic extensions, and their algebraic semantics. In Section 3, we introduce algebraic Kripke-style semantics for those systems, and connect them with algebraic semantics.

For convenience, we shall adopt the notation and terminology similar to those in Montagna & Sacchetti (2003; 2004) Yang (2016b; 2017c), and assume reader familiarity with them (together with results found therein).

2. Preliminaries: weakly associative fuzzy logics and their algebraic semantics

Here we briefly recall the systems and their algebraic semantics introduced in Yang (2014a) as preliminaries. Weakly associative fuzzy logics are based on a countable propositional language with formulas *Fm* built inductively as usual from a set of propositional variables *VAR*, binary connectives $\rightarrow$, &, $\wedge$, $\vee$, and constants **T**, **F**, **f**, **t**. Further definable connectives are:

df1. $\neg\phi := \phi \rightarrow \mathbf{f}$,
df2. $\phi \leftrightarrow \psi := (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$.

We may define **t** as $\mathbf{f} \rightarrow \mathbf{f}$. We moreover define $\phi^n_{\mathbf{t}}$ as $\phi_{\mathbf{t}}$ & $\cdots$ & $\phi_{\mathbf{t}}$, n factors, where $\phi_{\mathbf{t}} := \phi \wedge \mathbf{t}$. For the rest of this paper, we use the customary notations and terminology, and the axiom systems to provide a consequence relation.

We start with the following axiom schemes and rules for the weak **t**-associative monoidal uninorm logic $\mathbf{WA_tMUL}$ and its two axiomatic extensions.

Definition 2.1 (Yang (2016b)) (i) $\mathbf{WA_tMUL}$ consists of the following axiom schemes and rules:

A1. $\phi \rightarrow \phi$ (self-implication, SI)
A2. $(\phi \wedge \psi) \rightarrow \phi$, $(\phi \wedge \psi) \rightarrow \psi$ ($\wedge$-elimination, $\wedge$-E)
A3. $((\phi\rightarrow\psi)\wedge(\phi\rightarrow\chi)) \rightarrow (\phi\rightarrow(\psi\wedge\chi))$ ($\wedge$-introduction, $\wedge$-I)
A4. $\phi \rightarrow (\phi \vee \psi)$, $\psi \rightarrow (\phi \vee \psi)$ ($\vee$-introduction, $\vee$-I)
A5. $((\phi\rightarrow\chi)\wedge(\psi\rightarrow\chi)) \rightarrow ((\phi\vee\psi)\rightarrow\chi)$ ($\vee$-elimination, $\vee$-E)
A6. $\mathbf{F} \rightarrow \phi$ (ex falsum quodlibet, EF)
A7. $(\phi \,\&\, \psi) \rightarrow (\psi \,\&\, \phi)$ (&-commutativity, &-C)
A8. $\phi \leftrightarrow (\mathbf{t} \rightarrow \phi)$ (push and pop, PP)
A9. $\phi \rightarrow (\psi \rightarrow (\psi \,\&\, \phi))$ (&-adjunction, &-Adj)
A10. $(\phi_{\mathbf{t}} \,\&\, \psi_{\mathbf{t}}) \rightarrow (\phi \wedge \psi)$ (&$\wedge$)

A11. $(\psi \ \& \ (\phi \ \& \ (\phi \rightarrow (\psi \rightarrow \chi)))) \rightarrow \chi$ (residuation, Res')

A12. $(\phi \rightarrow ((\phi \ \& \ (\phi \rightarrow \psi)) \ \& \ (\psi \rightarrow \chi))) \rightarrow (\phi \rightarrow \chi)$ (T')

A13. $((\delta \& \varepsilon) \rightarrow (\delta \& (\varepsilon \& (\phi \rightarrow \psi)_t))) \vee (\delta' \rightarrow (\varepsilon' \rightarrow ((\varepsilon' \& \delta') \& (\psi \rightarrow \phi)_t)))$ (PL)

A14. $(\phi_t \& (\psi_t \& \chi_t)) \leftrightarrow ((\phi_t \& \psi_t) \& \chi_t)$ (weak **t**-associativity, wAS_t)

$\phi \rightarrow \psi, \phi \vdash \psi$ (modus ponens, mp)

$\phi \vdash \phi_t$ (adj_u)

$\phi \vdash (\delta \ \& \ \varepsilon) \rightarrow (\delta \ \& \ (\varepsilon \ \& \ \phi))$ (α)

$\phi \vdash \delta \rightarrow (\varepsilon \rightarrow ((\varepsilon \ \& \ \delta) \ \& \ \phi))$ (β).

(ii) The following are weakly associative fuzzy logics that axiomatically extend $\mathbf{WA_tMUL}$:

- **t**-associative (ta-) monoidal uninorm logic $\mathbf{A_tMUL}$ is $\mathbf{WA_tMUL}$ plus

(AS_t) $(\phi \ \& \ (\psi \ \& \ \chi))_t \leftrightarrow ((\phi \ \& \ \psi) \ \& \ \chi)_t$;

(RE_t) $(\phi \rightarrow (\psi \rightarrow \chi))_t \leftrightarrow ((\phi \ \& \ \psi) \rightarrow \chi)_t$;

(SF_t) $(\phi \rightarrow \psi)_t \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi))$;

(PF_t) $(\psi \rightarrow \chi)_t \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))$; and

(MT_t) $(\phi \rightarrow \psi)_t \rightarrow ((\phi \ \& \ \chi) \rightarrow (\psi \ \& \ \chi))$.

- Strong ta-monoidal uninorm logic $\mathbf{SA_tMUL}$ is $\mathbf{A_tMUL}$ plus

(sAS_t) $(\phi_t \ \& \ (\psi \ \& \ \chi)) \leftrightarrow ((\phi_t \ \& \ \psi) \ \& \ \chi)$.

For easy reference, we let Ls be a set of weakly associative fuzzy logics defined previously.

Definition 2.2 Ls = {$\mathbf{WA_tMUL}$, $\mathbf{A_tMUL}$, $\mathbf{SA_tMUL}$}.

In L ($\in$ Ls), **f** can be defined as $\neg$**t** and vice versa.

A *theory* over L ($\in$ Ls) is a set T of formulas. A *proof* in a sequence of formulas whose each member is either an axiom of L or a member of T or follows from some preceding members of the sequence using a rule of L. T $\vdash$ ϕ, more exactly T $\vdash_L$ ϕ, means that ϕ is *provable* in T w.r.t. L, i.e., there is an L-proof of ϕ in T. A theory T is *inconsistent* if T $\vdash$ **F**; otherwise it is *consistent*.

The deduction theorem for L is as follows:

Proposition 2.3 Let T be a theory, and ϕ, ψ formulas.

(i) (Cintula et al. (2013; 2015)) T $\cup$ $\{\phi\}$ $\vdash_L$ ψ iff T $\vdash_L$ $\gamma(\phi) \rightarrow \psi$ for some $\gamma \in \Pi(\mathrm{bDT}^*)$.[4)]

(ii) (Yang (2009)) For L $\in$ {**A$_t$MUL**, **SA$_t$MUL**}, T $\cup$ $\{\phi\}$ $\vdash_L$ ψ iff there is n such that T $\vdash_L$ $\phi^n{}_{\mathbf{t}} \rightarrow \psi$.

For convenience, "$\neg$", "$\wedge$", "$\vee$", and "$\rightarrow$" are used ambiguously as propositional connectives and as algebraic operators, but context should clarify their meanings.

Suitable algebraic structures for L ($\in$ Ls) are obtained as varieties of residuated lattice-ordered unital groupoids (briefly, rlu-groupoids) in the sense of Galatos et al. (2007).

Definition 2.4 (i) A *pointed bounded commutative rlu-groupoid* is a structure **A** = (A, $\top$, $\bot$, t, f, $\wedge$, $\vee$, *, $\rightarrow$) such that:

(I) (A, $\top$, $\bot$, $\wedge$, $\vee$) is a bounded lattice with top element

4) For γ and $\Pi(\mathrm{bDT}^*)$, see Cintula et al. (2013; 2015) and Yang (2015a).

$\top$ and bottom element $\bot$.

(Ⅱ) (A, *, t) is a commutative groupoid with unit.

(Ⅲ) $y \leq x \to z$ iff $x * y \leq z$, for all $x, y, z \in A$ (residuation).

(ii) (Yang (2015a)) An *MICAL-algebra* is a pointed bounded commutative rlu-groupoid satisfying: for all x, y, z, w, z', w' $\in$ A,

(PL^A) $t \leq ((z*w) \to (z*(w*(x\to y)_t))) \vee (z' \to (w'\to((w'*z')*(y \to x)_t)))$.

Definition 2.5 (L-algebras, Yang (2016a)) A *WA_tMUL-algebra* is an MICAL-algebra satisfying: (wAS_t^A) $x_t * (y_t * z_t) = (x_t * y_t) * z_t$, for all $x, y, z \in A$; an *A_tMUL-algebra* is an MICAL-algebra satisfying: for all $x, y, z \in A$, (AS_t^A) $(x * (y * z))_t = ((x * y) * z)_t$, ($RE_t^A$) $(x \to (y \to z))_t = ((x * y) \to z)_t$, ($SF_t^A$) $(x \to y)_t \leq ((y \to z) \to (x \to z))$, ($PF_t^A$) $(y \to z)_t \leq ((x \to y) \to (x \to z))$, and ($MT_t^A$) $(x \to y)_t \leq ((x * z) \to (y * z))$; an *$SA_tMUL$-algebra* is an A_tMUL-algebra satisfying: (sAS_t^A) $x_t * (y * z) = (x_t * y) * z$, for all $x, y, z \in A$. We call all these algebras *L-algebras*.

A commutative unital groupoid (A, *, t) satisfying (associativity) $x * (y * z) = (x * y) * z$ on [0, 1] is a *uninorm* and this is a *t-norm* in case $t = \top$.

By x^n, we denote $x * \cdots * x$, n factors. Using $\to$ and f we can define t as $f \to f$, and $\neg$ as in (df1).

For L ($\in$ Ls), an L-algebra is said to be *linearly ordered* if the ordering of its algebra is linear, i.e., $x \leq y$ or $y \leq x$ (equivalently, $x \wedge y = x$ or $x \wedge y = y$) for each pair x, y. Note that, if an L-algebra is linearly ordered, each algebra can be defined as follows: A *WA_tMUL-algebra* is an MICAL-algebra satisfying (wAS_t^{A}') $x * (y * z) = (x * y) * z$ if $x, y, z \leq t$; an *A_tMUL-algebra* is an MICAL-algebra satisfying (AS_t^{A}') $\min\{x * (y * z), t\} = \{(x * y) * z, t\}$ for all $x, y, z \in A$: and an *SA_tMUL-algebra* is an MICAL-algebra satisfying (sAS_t^{A}') $x * (y * z) = (x * y) * z$ if $x \leq t$ or $y \leq t$ or $z \leq t$.

Definition 2.6 (Evaluation) Let $\mathcal{A}$ be an algebra. An *$\mathcal{A}$-evaluation* is a function $v : FOR \to \mathcal{A}$ satisfying: $v(\phi \to \psi) = v(\phi) \to v(\psi)$, $v(\phi \wedge \psi) = v(\phi) \wedge v(\psi)$, $v(\phi \vee \psi) = v(\phi) \vee v(\psi)$, $v(\phi \,\&\, \psi) = v(\phi) * v(\psi)$, $v(\mathbf{T}) = \top$, $v(\mathbf{F}) = \bot$, $v(\mathbf{f}) = f$, (and hence $v(\neg\phi) = \neg v(\phi)$ and $v(\mathbf{t}) = t$).

Definition 2.7 Let $\mathcal{A}$ be an L-algebra, T be a theory, ϕ be a formula, and K be a class of L-algebras.

(i) (Tautology) ϕ is a *t-tautology* in $\mathcal{A}$, briefly an *$\mathcal{A}$-tautology* (or *$\mathcal{A}$-valid*), if $v(\phi) \geq t$ for each $\mathcal{A}$-evaluation v.

(ii) (Model) An $\mathcal{A}$-evaluation v is an *$\mathcal{A}$-model* of T if $v(\phi) \geq t$ for each $\phi \in T$. We denote the class of $\mathcal{A}$-models of T, by *Mod(T, $\mathcal{A}$)*.

(iii) (Semantic consequence) ϕ is a *semantic consequence* of T w.r.t. K, denoting by $T \vDash_{\mathsf{K}} \phi$, if $Mod(T, \mathcal{A}) = Mod(T \cup \{\phi\}, \mathcal{A})$ for each $\mathcal{A} \in \mathsf{K}$.

Definition 2.8 (**L**-algebra) Let $\mathcal{A}$, T, and ϕ be as in Definition 2.7. $\mathcal{A}$ is an ***L**-algebra* iff, whenever ϕ is L-provable in T (i.e. T $\vdash_L \phi$, L an L logic), it is a semantic consequence of T w.r.t. the set $\{\mathcal{A}\}$ (i.e. T $\vDash_{\{A\}} \phi$), $\mathcal{A}$ a corresponding L-algebra). By $MOD^{(l)}(L)$, we denote the class of (linearly ordered) **L**-algebras. Finally, we write T $\vDash^{(l)}_L \phi$ in place of T $\vDash_{MOD^{(l)}(L)} \phi$.

Theorem 2.9 (Strong completeness, Yang (2016b)) Let T be a theory, and ϕ a formula. T $\vdash_L \phi$ iff T $\vDash_L \phi$ iff T $\vDash^{l}_L \phi$.

Definition 2.10 An L-algebra is *standard* iff its lattice reduct is [0, 1].

Theorem 2.11 (Strong standard completeness, Yang (2016b)) For $\mathbf{WA_tMUL}$, the following are equivalent:

(1) T $\vdash_{\mathbf{WAtMUL}} \phi$.

(2) For every standard $\mathbf{WA_tMUL}$-algebra and evaluation v, if v$(\psi) \geq e$ for all $\psi \in$ T, then v$(\phi) \geq e$.

3. Algebraic Kripke-style semantics

Here, we consider *general* algebraic Kripke-style semantics for **UL** and its extensions.

Definition 3.1 (i) (Algebraic Kripke frame) An *algebraic Kripke frame* is a structure $\mathbf{X}$ = (X, $\top$, $\bot$, t, f, $\leq$, $*$, $\rightarrow$) such that (X, $\top$, $\bot$, t, f, $\leq$, $*$, $\rightarrow$) is a linearly ordered pointed bounded commutative rlu-groupoid. The elements of **X** are called *nodes*.

(ii) (MICAL frame) An *MICAL frame* is an algebraic Kripke frame, where $*$ is conjunctive (i.e., $\bot * \top = \bot$) and left-continuous (i.e., whenever $\sup\{x_i : i \in I\}$ exists, $x * \sup\{x_i : i \in I\} = \sup\{x * x_i : i \in I\}$), and so its residuum $\rightarrow$ is defined as $x \rightarrow y := \sup\{z: x * z \leq y\}$ for all $x, y \in X$.

Definition 3.2 (L frame) A *WA$_t$MUL frame* is an MICAL frame satisfying (wAS$_t^{A}$'); an *A$_t$MUL frame* is an MICAL frame satisfying (AS$_t^{A}$'); and an *SA$_t$MUL frame* is an A$_t$MUL frame satisfying (sAS$_t^{A}$'). We call all these frames *L frames*.

Definition 3.1 (ii) ensures that an MICAL frame has a supremum w.r.t. $*$, i.e., for every $x, y \in X$, the set $\{z: x * z \leq y\}$ has the supremum. **X** is said to be *complete* if $\leq$ is a complete order.

An *evaluation* or *forcing* on an algebraic Kripke frame is a relation $\Vdash$ between nodes and propositional variables, and arbitrary formulas subject to the conditions below: for every propositional variable p,

(AHC) if $x \Vdash p$ and $y \leq x$, then $y \Vdash p$;
(min) $\bot \Vdash p$; and

for arbitrary formulas,

(t) $x \Vdash \mathbf{t}$ iff $x \leq t$;

(f) $x \Vdash \mathbf{f}$ iff $x \leq f$;

($\perp$) $x \Vdash \mathbf{F}$ iff $x = \perp$;

($\wedge$) $x \Vdash \phi \wedge \psi$ iff $x \Vdash \phi$ and $x \Vdash \psi$;

($\vee$) $x \Vdash \phi \vee \psi$ iff $x \Vdash \phi$ or $x \Vdash \psi$;

(&) $x \Vdash \phi \,\&\, \psi$ iff there are $y, z \in X$ such that $y \Vdash \phi$, $z \Vdash \psi$, and $x \leq y * z$;

($\rightarrow$) $x \Vdash \phi \rightarrow \psi$ iff for all $y \in X$, if $y \Vdash \phi$, then $x * y \Vdash \psi$.

An evaluation or forcing on an L frame is an evaluation or forcing further satisfying that (max) for every atomic sentence p, $\{x : x \Vdash p\}$ has a maximum.

Definition 3.3 (i) (Algebraic Kripke model) An *algebraic Kripke model* is a pair $(\mathbf{X}, \Vdash)$, where $\mathbf{X}$ is an algebraic Kripke frame and $\Vdash$ is a forcing on $\mathbf{X}$.

(ii) (L model) An *L model* is a pair $(\mathbf{X}, \Vdash)$, where $\mathbf{X}$ is an L frame and $\Vdash$ is a forcing on $\mathbf{X}$. an L model $(\mathbf{X}, \Vdash)$ is said to be *complete* if $\mathbf{X}$ is a complete frame and $\Vdash$ is a forcing on $\mathbf{X}$.

Definition 3.4 Given an algebraic Kripke model $(\mathbf{X}, \Vdash)$, a node x of $\mathbf{X}$ and a formula ϕ, we say that *x forces ϕ* to express $x \Vdash \phi$. We say that ϕ is *true* in $(\mathbf{X}, \Vdash)$ if $t \Vdash \phi$, and that ϕ is *valid* in the frame $\mathbf{X}$ (expressed by $\mathbf{X} \vDash \phi$) if ϕ is true in $(\mathbf{X}, \Vdash)$ for every forcing $\Vdash$ on $\mathbf{X}$.

For soundness and completeness for L, let $\vdash_L \phi$ be the theoremhood of ϕ in L. First we can easily show the following lemma.

Lemma 3.5 (Cf, Yang (2016b)) (i) (Hereditary Lemma, HL) Let **X** be an algebraic Kripke frame. For any sentence ϕ and for all nodes x, y $\in$ **X**, if $x \Vdash \phi$ and $y \leq x$, then $y \Vdash \phi$.

(ii) Let $\Vdash$ be a forcing on an L frame, and ϕ a sentence. Then the set $\{x \in X : x \Vdash \phi\}$ has a maximum.

Proposition 3.6 (Soundness) If $\vdash_L \phi$, then ϕ is valid in every L frame.

Proof: We prove the validity of (wAS_t), (AS_t), and (sAS_t) as examples.

(wAS_t) We need to show that $t \Vdash (\phi_t\&(\psi_t\&\chi_t)) \leftrightarrow ((\phi_t\&\psi_t)\&\chi_t)$. For the left-to-right direction, it suffices to assume $x \Vdash \phi_t \& (\psi_t \& \chi_t)$ and show $x \Vdash (\phi_t \& \psi_t) \& \chi_t$. Let $x \Vdash \phi_t \& (\psi_t \& \chi_t)$. Using the condition (&) twice, we have $y \Vdash \phi_t$, $z \Vdash \psi_t \& \chi_t$, $x \leq y * z$, $v \Vdash \psi_t$, $w \Vdash \chi_t$, and $z \leq v * w$. Thus, we also have $x \leq y * z \leq y * (v * w)$. Then, since the conditions (t) and ($\wedge$) ensure that x, y, $z \leq t$, we have $y * (v * w) = (y * v) * w$ by ($wAS_t^{A'}$). Thus, using the condition (&) twice, we have $y * v \Vdash \phi_t \& \psi_t$ and $(y * v) * w \Vdash (\phi_t \& \psi_t) \& \chi_t$; therefore, $x \Vdash (\phi_t \& \psi_t) \& \chi_t$ since $x \leq y * (v * w) = (y * v) * w$. Analogously, we can prove the right-to-left direction.

(AS_t) We need to show that $t \Vdash (\phi\&(\psi\&\chi))_t \leftrightarrow ((\phi\&\psi)\&\chi)_t$. For the left-to-right direction, it suffices to assume $x \Vdash (\phi \& (\psi \& \chi))_t$ and show $x \Vdash ((\phi \& \psi) \& \chi)_t$. Let $x \Vdash (\phi \& (\psi \& \chi))_t$. Using the conditions (t) and ($\wedge$), we have $x \Vdash \phi \& (\psi \& \chi)$ and $x \leq t$. Then, using the condition (&) twice, we have $y \Vdash \phi$, $z \Vdash \psi \& \chi$, $x \leq y * z$, $v \Vdash \psi$, $w \Vdash \chi$, and $z \leq v * w$. Thus, we also have $x \leq \min\{t, y * z\} \leq \min\{t, y * (v * w)\}$. Then, we have $\min\{t, y * (v * w)\} = \min\{t, (y * v) * w\}$ by (AS_t^{A}'). Thus, using the condition (&) twice, we have $y * v \Vdash \phi \& \psi$ and $(y * v) * w \Vdash (\phi \& \psi) \& \chi$; therefore, using the conditions (t) and ($\wedge$), we obtain $x \Vdash ((\phi \& \psi) \& \chi)_t$ since $x \leq \min\{t, (y * v) * w\}$. Analogously, we can prove the right-to-left direction.

(sAS_t) We need to show that $t \Vdash \phi_t\&(\psi\&\chi) \leftrightarrow (\phi_t\&\psi)\&\chi$. For the left-to-right direction, it suffices to assume $x \Vdash \phi_t \& (\psi \& \chi)$ and show $x \Vdash (\phi_t \& \psi) \& \chi$. Let $x \Vdash \phi_t \& (\psi \& \chi)$. Using the condition (&) twice, we have $y \Vdash \phi_t$, $z \Vdash \psi \& \chi$, $x \leq y * z$, $v \Vdash \psi$, $w \Vdash \chi$, and $z \leq v * w$. Thus, we also have $x \leq y * z \leq y * (v * w)$. Then, since the conditions (t) and ($\wedge$) ensure that $y \leq t$, we have $y * (v * w) = (y * v) * w$ by (sAS_t^{A}'). Thus, using the condition (&) twice, we have $y * v \Vdash \phi_t \& \psi$ and $(y * v) * w \Vdash (\phi_t \& \psi) \& \chi$; therefore, $x \Vdash (\phi_t \& \psi) \& \chi$ since $x \leq y * (v * w) = (y * v) * w$. Analogously, we can prove the right-to-left direction.

The proof for the other cases is left to the interested reader. □

By a *chain*, we mean a linearly ordered algebra. The next proposition connects algebraic Kripke semantics and algebraic semantics for L (cf. see Montagna & Sacchetti (2004)).

Proposition 3.7 (i) The $\{\top, \bot, t, f, \leq, *, \rightarrow\}$ reduct of an L-chain A is an L frame, which is complete iff A is complete.

(ii) Let $\mathbf{X} = (X, \top, \bot, t, f, \leq, *, \rightarrow)$ be an L frame. Then the structure $\mathbf{A} = (X, \top, \bot, t, f, max, min, *, \rightarrow)$ is an L-algebra (where *max* and *min* are meant w.r.t. $\leq$).

(iii) Let $\mathbf{X}$ be the $\{\top, \bot, t, f, \leq, *, \rightarrow\}$ reduct of an L-chain A, and let v be an evaluation in A. Let for every atomic formula p and for every $x \in A$, $x \Vdash p$ iff $x \leq v(p)$. Then $(\mathbf{X}, \Vdash)$ is an L model, and for every formula ϕ and for every $x \in A$, we obtain that: $x \Vdash \phi$ iff $x \leq v(\phi)$.

(iv) Let $(\mathbf{X}, \Vdash)$ be an L model, and let A be the L-algebra defined as in (ii). Define for every atomic formula p, $v(p) = max\{x \in X : x \Vdash p\}$. Then for every formula ϕ, $v(\phi) = max\{x \in X : x \Vdash \phi\}$.

Proof: The proof for (i) and (ii) is easy. Since (iv) follows almost directly from (iii) and Lemma 3.6 (ii), we prove (iii). As regards to claim (iii), we consider the induction steps corresponding to the cases where $\phi = \psi \,\&\, \chi$ and $\phi = \psi \rightarrow \chi$. (The proof for the other cases are trivial.)

Suppose $\phi = \psi \,\&\, \chi$. By the condition (&), $x \Vdash \psi \,\&\, \chi$ iff there are $y, z \in X$ such that $y \Vdash \psi$, $z \Vdash \chi$, and $x \leq y * z$, hence by the induction hypothesis, $y \Vdash \psi$ and $z \Vdash \chi$ iff $y \leq$

$v(\psi)$ and $z \leq v(\chi)$. Then, it holds true that $x \leq y * z \leq v(\psi) * v(\chi) = v(\psi \& \chi)$. Conversely, if $x \leq v(\psi) * v(\chi) = v(\psi \& \chi)$, then take $y = v(\psi)$ and $z = v(\chi)$. Then we have $x \leq y * z$, $y \Vdash \psi$, and $z \Vdash \chi$, therefore $x \Vdash \psi \& \chi$.

Suppose $\phi = \psi \to \chi$. By the condition ($\to$), $x \Vdash \psi \to \chi$ iff for all $y \in X$, if $y \Vdash \psi$, then $x * y \Vdash \chi$, hence by the induction hypothesis, $y \Vdash \psi$ only if $x * y \Vdash \chi$ iff $y \leq v(\psi)$ only if $x * y \leq v(\chi)$, therefore iff $x * v(\psi) \leq v(\chi)$, therefore by residuation, iff $x \leq v(\psi) \to v(\chi) = v(\psi \to \chi)$, as desired. □

Theorem 3.8 (Strong completeness)

(i) **L** is strongly complete w.r.t. the class of all L-frames.

(ii) **WA$_t$MUL**, is strongly complete w.r.t. the class of complete L-frames.

Proof: (i) and (ii) follow from Proposition 3.7 and Theorem 2.9, and from Proposition 3.7 and Theorem 2.11, respectively. □

4. Concluding remark

We investigated here just algebraic Kripke-style semantics for some weakly associative fuzzy logics. Note that, while Yang provided algebraic semantics for other non-associative fuzzy logics in Yang (2015a; 2016c, 2017a, 2017c). we did not consider algebraic Kripke-style semantics for those systems. To provide such semantics for these logics remains a problem to be solved.

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약한 결합 원리를 갖는 퍼지 논리를 위한 대수적 크립키형 의미론

양 은 석

이 글에서 우리는 (곱 연언 &의) 약한 형식의 결합 원리를 갖는 퍼지 논리를 위한 대수적 크립키형 의미론을 연구한다. 이를 위하여 먼저 약한 결합 원리를 갖는 퍼지 논리의 대수적 의미론을 소개한다. 다음으로 이 체계들을 위한 대수적 크립키형 의미론을 제공한 후, 이를 대수적 의미론과 연관 짓는다.

주요어: (대수적) 크립키형 의미론, 약화 없는 퍼지 논리, 약한 결합 원리, 대수적 의미론, 준구조 논리.