논리연구 24-1(2021) pp. 1-30

Tennant's Conjecture for Self-Referential Paradoxes and its Classical Counterexample*

Seungrak Choi

【Abstract】In his paper, "On paradox without self-reference", Neil Tennant proposed the conjecture for self-referential paradoxes that any derivation formalizing self-referential paradoxes only generates a looping reduction sequence. According to him, the derivation of the Liar paradox in natural deduction initiates a looping reduction sequence and the derivation of the Yablo's paradox generates a spiral reduction.

The present paper proposes the counterexample to Tennant's conjecture for self-referential paradoxes. We shall show that there is a derivation of the Liar paradox which generates a spiraling reduction procedure. Since the Liar paradox is a self-referential paradox, the result is a counterexample to his conjecture.

Tennant has believed that *classical reductio* has no essential role to formalize paradoxes. As our counterexample applies the rule of *classical reductio*, he may reject the counterexample. In this sense, it will be briefly argued that *classical reductio* and his rules for the liar sentence share some inferential role. If *classical reductio* should not be used in paradoxical reasoning, neither should be his rules for the liar sentence.

【Key Words】The liar paradox, Yablo's paradox, Self-reference, Classical reductio, Gunnar Stålmarck, Neil Tennant.

Received: Jan. 15, 2021. Revised: Feb. 18, 2021. Accepted: Feb. 16, 2021.

* This research has been supported by the AMOREPACIFIC Foundation. An early version of this paper was presented at the 1st Korea Logic Day. I would like to thank anonymous referees for their helpful comments and encouragement.

1 Introduction

While paradoxes are considered to be an inference, which inferential feature is of paradoxes? Attempts have been made to find certain feature from the proof-theoretic perspective. Natural deduction introduced by Gerhard Gentzen (1935) is one of the main tools of proof theory. Dag Prawitz (1965, p. 95) investigated Russell's paradox in natural deduction and discovered that the derivation formalizaing Russell's paradox falls into a non-terminating reduction sequence. Neil Tennant (1982, 1995, 2015, 2016, 2017) has regarded the non-terminating reduction sequence as the primary feature of paradoxes and proposed the proof-theoretic criterion for paradoxicality.

Tennant (1982) sets his criterion for paradoxicality such that the derivations formalizing paradoxes in natural deduction are distinguished by having non-terminating reduction sequences of the derivation of an absurdity ($\perp$) involved. He called a non-terminating reduction sequence a 'looping reduction sequence.' While investigating a non-self-referential paradox suggested by Stephen Yablo (1993), Tennant (1995) has extended his criterion by embracing a spiraling reduction generated by Yablo's paradox. He thought that a looping reduction sequence is the main feature of the self-referential paradoxes and Yablo's paradox is not a self-referential one. He claimed that the non-terminating reduction sequence enters into loops if the self-reference is involved; otherwise it does not. So to speak, derivations formalizing self-referential paradoxes produce only a looping reduction sequence. Thus, we interpret his claim as an informal conjecture for self-referential paradoxes that every derivation formalizing a self-referential paradox in natural deduction generates a looping reduction sequence but not a spiraling reduction sequence.

The present paper proposes a counterexample to Tennant's conjecture for self-referential paradoxes. After we introduce preliminary notations and Tennant's criterion for paradoxicality in Section 2, we deals with problems of Tennant's conjecture in Section 3, especially the counterexample to his. For the minor problem of his conjecture, it will be asked whether Tennant's spiraling reduction is distinct from his looping reduction sequence. With respect to the main problem of his, we first borrow the rules of the Liar paradox from Tennant (2016, 2017) and show that there is a derivation of the Liar paradox initiating a spiraling reduction sequence. The result can be a counterexample to his conjecture, in that the Liar paradox is a self-referential paradox but its derivation generates a spiraling reduction sequence.

The rule of *classical reductio*, i.e. *CR*−rule, is involved in our counterexample to Tennant's conjecture for self-referential paradoxes. Tennant (2017, pp. 281-285) has believed that *classical reductio* has no essential role in the derivation of paradoxes in the sense that no derivation of paradoxes need to use *classical reductio*. As the counterexample uses *CR*−rule, he would not admit the counterexample. In Section 4, we, however, argues that it is unconvincing that *clssical reductio* has no essential role in paradoxical reasoning, and hence the counterexample is acceptable.

2 Tennant's Conjecture for Self-Referential Paradoxes: Looping and Spiraling Reduction Sequences

Tennant (1995, p. 207) claimed that the distinguising feature of self-referential paradoxes is the non-terminating reduction procedures entering into loops. After we have preliminary notions and rules in Section 2.1, Tennant's criterion for paradoxicality and his conjecture for self-referential paradoxes will be introduced in Section 2.2.

2.1 Preliminaries: Some Terminologies and Natural Deduction Rules

Our language consists of symbols for arithmetic, i.e. the constant 0, the unary function symbol s for successors, the binary function symbol $+$ for addition, the binary predicate $<$ for less-than-relation, and $=$ for equality. Moreover, it has logical operators $\rightarrow$, $\bot$, and $\forall$ for implication, absurdity, and universal quantification respectively. We use u, v, w, x, y, z for free variables and the others for closed terms, and φ, ψ, and σ for arbitrary formula. A negation formula $\neg\varphi$ is defined by $\varphi \rightarrow \bot$.

Let $\mathfrak{D}$ be a derivation of a given natural deduction system, used in the same manner as 'deduction' in Prawitz (1965). Following Prawitz, we shall use the following conventions: if a derivation $\mathfrak{D}$ ends with a formula φ, we shall write $\begin{array}{c}\mathfrak{D}\\ \varphi\end{array}$ and φ is called an 'end-formula.' If it depends on a formula ψ, we shall write $\begin{array}{c}\psi\\ \mathfrak{D}\\ \varphi\end{array}$. Then, we have rules for $\rightarrow$, $\forall$, and *classical reductio* in the natural deduction style proposed by Prawitz (1965).

$$
\dfrac{\begin{array}{c}[\varphi]^1\\ \mathfrak{D}_1\\ \psi\end{array}}{\varphi \rightarrow \psi}\rightarrow I_{,1}
\qquad
\dfrac{\varphi \rightarrow \psi \quad \begin{array}{c}\mathfrak{D}_2\\ \varphi\end{array}}{\psi}\rightarrow E
\qquad
\dfrac{\begin{array}{c}\mathfrak{D}_1\\ \varphi[y/x]\end{array}}{\forall x\varphi(x)}\forall I
\qquad
\dfrac{\forall x\varphi(x)}{\varphi[t/x]}\forall E
\qquad
\dfrac{\begin{array}{c}[\neg\varphi]^1\\ \mathfrak{D}\\ \bot\end{array}}{\varphi}CR_{,1}
$$

$\varphi[x/y]$ means the substitution of x for y in φ. We call the formulas directly above the line in each rule, 'premises,' and the formula directly below the line, 'conclusion.' *Assumptions* which can be discharged are in the square brackets, e.g. $[\varphi]$. A *major premise* of the elimination rule for an operator is the premise containing the constant in the elimination rule and all other premises are *minor premises*. The *maximum formula* is the conclusion of an introduction rule (or of $CR-$rule) at the same time the major premise of an elimination rule. We accept the standard reduction procedures of Prawitz (1965, pp. 36-38).

Let us consider any two derivations $\mathfrak{D}_1$ and $\mathfrak{D}_2$ having the same end-formula. We say that a derivation $\mathfrak{D}'$ is an *immediate subderivation* of $\mathfrak{D}_1$ if $\mathfrak{D}'$ is an initial part of $\mathfrak{D}_1$ ending with a premise of the last inference step in $\mathfrak{D}_1$. Let $\mathfrak{D}_1 \rhd \mathfrak{D}_2$ mean that $\mathfrak{D}_1$ *reduces* to $\mathfrak{D}_2$ by applying a single reduction step to an immediate subderivation $\mathfrak{D}'$ of $\mathfrak{D}_1$. We will introduce standard reduction procedures for $\rightarrow$ and $\forall$ as follows.

$$\dfrac{\dfrac{\begin{array}{c}[\varphi]^1\\ \mathfrak{D}_1\\ \psi\end{array}}{\varphi \rightarrow \psi}{\rightarrow I_{,1}} \qquad \begin{array}{c}\mathfrak{D}_2\\ \varphi\end{array}}{\psi}{\rightarrow E} \quad \rhd_{\rightarrow} \quad \begin{array}{c}\mathfrak{D}_2\\ \varphi\\ \mathfrak{D}_1\\ \psi\end{array} \qquad\qquad \dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_1\\ \varphi(y)\end{array}}{\forall x\varphi[x/y]}{\forall I}}{\varphi[t/x]}{\forall E} \quad \rhd_{\forall} \quad \begin{array}{c}\mathfrak{D}_1\\ \varphi[t/y]\end{array}$$

The main role of these standard reduction procedures is to eliminate the maximum formula. When the derivation has no maximum formula, we say that it is in *normal form*. Let $\mathbb{R}$ be a set of reduction procedures. Every reduction procedure in $\mathbb{R}$ is to be closed under substitution of derivations for open assumptions, and the notions of 'normal' and 'normalizable' are defined in the following ways[1]

Definition 2.1. A sequence $<\mathfrak{D}_1, ..., \mathfrak{D}_i, \mathfrak{D}_{i+1}, ...>$ of derivations is a *reduction sequence* relative to $\mathbb{R}$ iff $\mathfrak{D}_i \rhd \mathfrak{D}_{i+1}$ relative to $\mathbb{R}$ where $1 \leqslant i$ for any natural number i. A derivation $\mathfrak{D}_1$ is *reducible* to $\mathfrak{D}_i$ relative to $\mathbb{R}$ iff there is a sequence $<\mathfrak{D}_1, \mathfrak{D}_2, ..., \mathfrak{D}_i>$ relative to $\mathbb{R}$ where for each $j < i$, $\mathfrak{D}_j \rhd \mathfrak{D}_{j+1}$; $\mathfrak{D}_1$ is *irreducible* relative to $\mathbb{R}$ iff there is no derivation $\mathfrak{D}'$ to which $\mathfrak{D}_1 \rhd \mathfrak{D}'$ relative to $\mathbb{R}$ except $\mathfrak{D}_1$ itself.

Definition 2.2. A derivation $\mathfrak{D}$ is *normal* (or in *normal form*) relative to $\mathbb{R}$ iff $\mathfrak{D}$ is irreducible relative to $\mathbb{R}$, i.e. $\mathfrak{D}$ has no maximum formula. A reduction sequence *terminates* iff it has a finite number of derivations and its last derivation is in normal form. A derivation $\mathfrak{D}$ is *normalizable* relative to $\mathbb{R}$ iff there is a terminating reduction sequence relative to $\mathbb{R}$ starting from $\mathfrak{D}$.

In the next subsection, we will take a derivation of the Liar paradox introduced in Tennant (2016, 2017) and of Yablo's paradox in Tennant (1995). We will see that both derivations generate different types of non-terminating reduction sequences, called looping and spiraling reductions, and so they are not normalizable. Tennant's criterion for paradoxicality will be introduced with respect to two types of infinite reduction sequences.

2.2 Tennant's Criterion for Paradoxicality

According to Tennant (1995), the Liar paradox and Yablo's paradox formalized in natural deduction generate non-terminating reduction

[1] In Definition 2.1, for any term x and y, let $x \leqslant y$ mean that x is less than or equal to y. For Definition 2.2, for our convenience sake, we drop the 'relative to $\mathbb{R}$ in the suggested notions if there is no misunderstanding.

sequences. However, they are distinguished in that the non-terminating reduction of the Liar enters into loops but that of Yablo's leads to a spiraling reduction. In order to introduce Tennant's criterion for paradoxicality and his distinction between self-referential and non-self-referential paradoxes, we shall consider two natural deduction systems S_L and S_Y for the Liar paradox and Yablo's paradox. First, both systems have rules for $\to$ and the following rules for an unary truth-predicate $T(x)$ which states that x is true.[2]

$$\frac{\varphi}{T(\ulcorner\varphi\urcorner)}\,TI \qquad \frac{T(\ulcorner\varphi\urcorner)}{\varphi}\,TE$$

The standard reduction procedure for $T(x)$ is as below.

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}\\ \varphi\end{array}}{T(\ulcorner\varphi\urcorner)}\,TI}{\varphi}\,TE \qquad \rhd_{T(x)} \qquad \begin{array}{c}\mathfrak{D}\\ \varphi\end{array}$$

For formulating the Liar paradox, S_L has a Tennant's rules for the liar sentence Φ from Tennant (2016, 2017).

[2] In order to apply truth predicate $T(x)$ to formulas, we use the left and right corner quotes, $\ulcorner\urcorner$. Let the function $\ulcorner - \urcorner$ be any injective mapping from formulas into expressions (or coded numerals). If coded numerals are considered, we borrow coding processes from Dirk van Dalen (2013, pp. 245-250). $\ulcorner - \urcorner$ refers (or encodes) to the expressions in our language. For any formula φ, $\ulcorner\varphi\urcorner$ refers to φ. If $\psi(x)$ is a formula with one free variable x, then $\psi(\ulcorner\varphi\urcorner)$ is a formula describing that a formula φ denoted by $\ulcorner\varphi\urcorner$ is ψ.

$$
\dfrac{\begin{matrix}[T(\ulcorner\Phi\urcorner)]^1\\ \mathfrak{D}_1\\ \bot\end{matrix}}{\Phi}\,\Phi I_{,1}
\qquad
\dfrac{\Phi \qquad \begin{matrix}[\neg T(\ulcorner\Phi\urcorner)]^1\\ \mathfrak{D}_2\\ \varphi\end{matrix}}{\varphi}\,\Phi E_{,1}
$$

Tennant (2016) proposes the reduction procedure for Φ as follows.

$$
\dfrac{\dfrac{\begin{matrix}[T(\ulcorner\Phi\urcorner)]^1\\ \mathfrak{D}_1\\ \bot\end{matrix}}{\Phi}\,\Phi I_{,1} \qquad \begin{matrix}[\neg T(\ulcorner\Phi\urcorner)]^2\\ \mathfrak{D}_2\\ \varphi\end{matrix}}{\varphi}\,\Phi E_{,2}
\quad \rhd_{\Phi} \quad
\begin{matrix}\dfrac{\begin{matrix}[T(\ulcorner\Phi\urcorner)]^1\\ \mathfrak{D}_1\\ \bot\end{matrix}}{\neg T(\ulcorner\Phi\urcorner)}\rightarrow I_{,1}\\ \mathfrak{D}_2\\ \varphi\end{matrix}
$$

S_L has a set $\mathbb{R}_L$ of reduction procedures including reductions for $\rightarrow$, $T(x)$, and Φ.

For formulating Yablo's paradox, S_Y does not have rules for Φ but uses the rules for $\forall$ and arithmetical axioms (or inferences).[3] S_Y has a set $\mathbb{R}_Y$ of reduction procedures containing reductions for $\rightarrow$, $\forall$, and $T(x)$.

In his recent work, Tennant (2016, 2017) put forward a derivation of the Liar paradox with the application of the rules for the liar sentence Φ. For our convenience sake, we deliver his proof in S_L.

Proposition 2.3. *There is a closed derivation of* $\bot$ *in* S_L *with respect to* $\mathbb{R}_L$*, which initiates a non-terminating reduction sequence and so*

[3]When we deal with Yablo's paradox in S_Y, we use the axioms of Peano arithmetic. For the reader who prefers to use arithmetical inferences in natural deduction, we borrow each form of arithmetical inference from Jan von Plato (2013, pp. 179-182).

is not normalizable.

Proof. We begin with the proof of $\bot$ and show that it fails to reduce a normal derivation.

Claim 1. There is a closed derivation Σ_3 of $\bot$.

First, we have an open derivation Σ_1 of $\bot$ from $[T(\ulcorner\Phi\urcorner)]$ below left. With Σ_1, there is a closed derivation Σ_2 of $T(\ulcorner\Phi\urcorner)$ below right.

$$\dfrac{\dfrac{[T(\ulcorner\Phi\urcorner)]^1}{\Phi}\,TE \qquad \dfrac{[\neg T(\ulcorner\Phi\urcorner)]^2 \quad [T(\ulcorner\Phi\urcorner)]^1}{\bot}\to E}{\bot}\,\Phi E_{,2} \qquad\qquad \dfrac{\dfrac{\begin{matrix}[T(\ulcorner\Phi\urcorner)]^1\\ \Sigma_1\\ \bot\end{matrix}}{\Phi}\,\Phi I_{,1}}{T(\ulcorner\Phi\urcorner)}\,TI$$

Then, we have a closed derivation Σ_3 of $\bot$.

$$\dfrac{\dfrac{\begin{matrix}[T(\ulcorner\Phi\urcorner)]^1\\ \Sigma_1\\ \bot\end{matrix}}{\neg T(\ulcorner\Phi\urcorner)}\to I_{,1} \qquad \begin{matrix}\Sigma_2\\ T(\ulcorner\Phi\urcorner)\end{matrix}}{\bot}\to E$$

Claim 2. Σ_3 generates a non-terminating reduction sequence and so is not normalizable.

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}[T(\ulcorner\Phi\urcorner)]^1\\ \Sigma_1\\ \bot\end{matrix}}{\Phi}\,\Phi I_{,1}}{T(\ulcorner\Phi\urcorner)}\,TI}{\Phi}\,TE \qquad \dfrac{[\neg T(\ulcorner\Phi\urcorner)]^2 \quad \dfrac{\dfrac{\begin{matrix}[T(\ulcorner\Phi\urcorner)]^3\\ \Sigma_1\\ \bot\end{matrix}}{\Phi}\,\Phi I_{,3}}{T(\ulcorner\Phi\urcorner)}\,TI}{\bot}\to E}{\bot}\,\Phi E_{,2}$$

$\neg T(\ulcorner\Phi\urcorner)$ in the last $\rightarrow E-$rule of Σ_3 is a maximum formula. Σ_3 reduces to the derivation Σ_4 above. By applying $\triangleright_{T(x)}$ and $\triangleright_{\Phi}$ to Σ_4, we have the same derivation with Σ_3. Σ_3 generates a non-terminating reduction sequence and thus it is not normalizable. □

The reduction procedure of Σ_3 ends up oscillating infinitely between three reductions, such as $\triangleright_{\rightarrow}$, $\triangleright_{T(x)}$, and $\triangleright_{\Phi}$. The reduction process cannot eliminate every maximum formula because it yields maximum formulas, i.e. $\neg T(\ulcorner\Phi\urcorner)$, $T(\ulcorner\Phi\urcorner)$, and Φ. Tennant (1982, 1995) has described this phenomenon as *falling into a looping reduction sequence.* He has regarded the non-terminating reduction sequence as the distinguishing feature of paradoxes, as Tennant (2016) summarized,

> Tennant (1982) proposed a proof-theoretic criterion, or test, for paradoxicality – that of *non-terminating reduction sequence* initiated by the 'proofs of $\perp$' associated with the paradoxes in question (p. 271).

Another key frature of paradoxes is that their derivations in natural deduction allow the form of inferences from φ to $\neg\varphi$ and $\neg\varphi$ to φ which is what Tennant (1982, p. 271) called *id est* inferences. The id est inferences may be any inferences having a formula interdeducible with its own negation (or its predication). For instance, the derivation Σ_3 of Proposition 2.3 uses $\Phi I-$ and $\Phi E-$rules, and Tennant (2016) called $\Phi-$rules the id est rules.

> [$\Phi I-$ and $\Phi E-$rules] are the 'id est' rules for the Liar (so-called because of the familiar transitions '[Φ], i.e. [$\neg T(\ulcorner\Phi\urcorner)$]'). The rule [$\Phi$ I- and $\Phi E-rules$] ensure that the sentence called [Φ] is *interdeducible with* [$\neg T(\ulcorner\Phi\urcorner)$].

We summarize his criterion as the proof-theoretic criterion for paradoxicality.[4]

Tennant's Criterion for Paradoxicality(*TCP*): Let $\mathfrak{D}$ be any derivation of a given natural deduction system S. $\mathfrak{D}$ is a *T-paradox* if and only if

(i) $\mathfrak{D}$ is a (closed or open) derivation of $\perp$ [5],

(ii) *id est* inferences (or rules) are used in $\mathfrak{D}$,

(iii) a reduction procedure of $\mathfrak{D}$ generates a non-terminating reduction sequence, such as a reduction loop.

Tennant (2016, 2017) has suggested fourth condition that all elimination rules are to be stated in generalized form in order to solve the problem proposed by Schroeder-Heister and Tranchini (2017). They have suggested Ekman's paradox taken from Jan Ekman (1998) to show that *TCP* overgenerates in the sense that there exists a derivation which is intuitively non-paradoxical but satisfies *TCP*. Unfortunately, Tennant's fourth condition fails to solve the problem. Schroeder-Heister and Tranchini (2018) have argued that *TCP* plus fourth condition overgenerates again by showing that there is a derivation of Ekman's paradox which is stated in generalized form but satisfies *TCP* with fourth condition. Therefore, we set aside Tennant's fourth condition for *TCP*.

---

[4] Tennant (2017, p. 288) has regarded *TCP* as a conjecture for genuine paradoxes. While considering *TCP* to be the conjecture for genuine paradoxes, we assume that, for any derivation $\mathfrak{D}$, $\mathfrak{D}$ formalizes a genuine paradox iff $\mathfrak{D}$ is a T-paradox.

[5] $\perp$ is not the only unacceptable conclusion. We can use a propositional variable p as an unacceptable conclusion while formulating Curry's paradox. For the examination of other cases, the reader can consult Tennant (1982)

2.3 Tennant on Yablo's Paradox

Now, we introduce a spiraling reduction sequence. Tennant (1995) extended *TCP* by embracing a spiraling reduction sequence occurred by a derivation formalizing Yablo's paradox. Stephen Yablo (1993) gave a liar-type paradox that, he claimed, avoids self-reference. Tennant (1995) accepted Yablo's paradox as a non-self-referential paradox and claimed that a derivation of a non-self-referential paradox does not generate a looping reduction sequence but a spiraling reduction sequence.

Yablo's paradox begins with an infinite sequence of sentences $S_1, S_2, S_3, \ldots$, each to the effect that every subsequent sentence is not true.

(S_1) for all $u > 1$, S_u is not true,

(S_2) for all $u > 2$, S_u is not true,

(S_3) for all $u > 3$, S_u is not true, ...

For the formalization of Yablo's paradox in natural deduction system S_Y, we define S_v as $\forall x(x > v \rightarrow \neg T(\ulcorner S_x \urcorner))$ and the following arithmetical facts.

Arithmetical Fact 1. **(**$AF1$**)** $\forall x \forall y(x > y+1 \rightarrow x > y)$.

Arithmetical Fact 2. **(**$AF2$**)** $\forall x(x+1 > x)$.

Also, for convenience, we use two abbreviations below.

$$\dfrac{\begin{array}{c}\mathfrak{D}\\ S_w\end{array}}{\neg T(\ulcorner S_{w+1} \urcorner)}\, Ab_{\neg T}(w) \qquad \dfrac{\begin{array}{c}\mathfrak{D}\\ S_w\end{array}}{S_{w+1}}\, Ab_S(w)$$

The left form $Ab_{\neg T}(w)$ above is an abbreviation for

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathfrak{D}\\ S_w\end{matrix}}{\forall x(x>w\to\neg T(\ulcorner S_{w+1}\urcorner))}\,def}{w+1>w\to\neg T(\ulcorner S_{w+1}\urcorner)}\,\forall E \qquad \dfrac{\dfrac{}{\forall x(x+1>x)}\,AF1}{w+1>w}\,\forall E}{\neg T(\ulcorner S_{w+1}\urcorner)}\,\to E$$

The right form $Ab_S(w)$ above is an abbreviation for

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\mathfrak{D}\\ S_w\end{matrix}}{\forall x(x>w\to\neg T(\ulcorner S_x\urcorner))}\,def}{u>w\to\neg T(\ulcorner S_u\urcorner)}\,\forall E \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\forall x\forall y(x>y+1\to x>y)}\,AF2}{\forall y(u>y+1\to u>y)}\,\forall E}{u>w+1\to u>w}\,\forall E \quad [u>w+1]^2}{u>w}\,\to E}{\neg T(\ulcorner S_u\urcorner)}\,\to E}{u>w+1\to\neg T(\ulcorner S_u\urcorner)}\,\to I_{,2}}{\dfrac{\forall x(x>w+1\to\neg T(\ulcorner S_x\urcorner))}{S_{w+1}}\,def}\,\forall I$$

Then, the following result describes Yablo's paradox.

Proposition 2.4. *There is a closed derivation of $\bot$ in S_Y with respect to $\mathbb{R}_Y$, which initiates a non-terminating reduction sequence and so is not normalizable.*

Proof. Two claims prove the result.

Claim 1. There is a closed derivation $\Delta_1(v)$ of $\bot$.

At first, we have an open derivation $\theta(w)$ of $\bot$ from $[T(\ulcorner S_w\urcorner)]$ as below left. Then, with $\theta(w)$, we have a closed derivation $\Delta_1(v)$ of

$\bot$ as below right.

$$
\begin{prooftree}
\AxiomC{$[T(\ulcorner S_w \urcorner)]^1$}
\RightLabel{TE}
\UnaryInfC{S_w}
\doubleLine
\RightLabel{$Ab_{\neg T}(w)$}
\UnaryInfC{$\neg T(\ulcorner S_{w+1} \urcorner)$}
\AxiomC{$[T(\ulcorner S_w \urcorner)]^1$}
\RightLabel{TE}
\UnaryInfC{S_w}
\doubleLine
\RightLabel{$Ab_s(w)$}
\UnaryInfC{S_{w+1}}
\RightLabel{TI}
\UnaryInfC{$T(\ulcorner S_{w+1} \urcorner)$}
\RightLabel{$\to E$}
\BinaryInfC{$\bot$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$[T(\ulcorner S_w \urcorner)]^1$}
\noLine
\UnaryInfC{$\theta(w)$}
\noLine
\UnaryInfC{$\bot$}
\RightLabel{$\to I_{,1}$}
\UnaryInfC{$\neg T(\ulcorner S_w \urcorner)$}
\RightLabel{$\to I_{,\emptyset}$}
\UnaryInfC{$w > v \to \neg T(\ulcorner S_w \urcorner)$}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall x(x > v \to \neg T(\ulcorner S_x \urcorner))$}
\dottedLine
\RightLabel{def}
\UnaryInfC{S_v}
\RightLabel{TI}
\UnaryInfC{$T(\ulcorner S_v \urcorner)$}
\noLine
\UnaryInfC{$\theta(v)$}
\noLine
\UnaryInfC{$\bot$}
\end{prooftree}
$$

Claim 2. $\Delta_1(v)$ generates a non-terminating reduction sequence and so is not normalizable.

$$
\begin{prooftree}
\AxiomC{$[T(\ulcorner S_w \urcorner)]^1$}
\noLine
\UnaryInfC{$\theta(w)$}
\noLine
\UnaryInfC{$\bot$}
\RightLabel{$\to I_{,1}$}
\UnaryInfC{$\neg T(\ulcorner S_w \urcorner)$}
\RightLabel{$\to I_{,\emptyset}$}
\UnaryInfC{$w > v \to \neg T(\ulcorner S_w \urcorner)$}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall x(x > v \to \neg T(\ulcorner S_x \urcorner))$}
\RightLabel{$\forall E$}
\UnaryInfC{$v+1 > v \to \neg T(\ulcorner S_{v+1} \urcorner)$}
\AxiomC{}
\RightLabel{$AF1$}
\UnaryInfC{$\forall x(x+1 > x)$}
\RightLabel{$\forall E$}
\UnaryInfC{$v+1 > v$}
\RightLabel{$\to E$}
\BinaryInfC{$\neg T(\ulcorner S_{v+1} \urcorner)$}
\AxiomC{$[T(\ulcorner S_w \urcorner)]^2$}
\noLine
\UnaryInfC{$\theta(w)$}
\noLine
\UnaryInfC{$\bot$}
\RightLabel{$\to I_{,2}$}
\UnaryInfC{$\neg T(\ulcorner S_w \urcorner)$}
\RightLabel{$\to I_{,\emptyset}$}
\UnaryInfC{$w > v \to \neg T(\ulcorner S_w \urcorner)$}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall x(x > v \to \neg T(\ulcorner S_x \urcorner))$}
\RightLabel{$\forall E$}
\UnaryInfC{$w > v \to \neg T(\ulcorner S_w \urcorner)$}
\AxiomC{$[w > v+1]^3$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$w > v$}
\RightLabel{$\to E$}
\BinaryInfC{$\neg T(\ulcorner S_w \urcorner)$}
\RightLabel{$\to I_{,3}$}
\UnaryInfC{$w > v+1 \to \neg T(\ulcorner S_w \urcorner)$}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall x(x > v+1 \to \neg T(\ulcorner S_x \urcorner))$}
\dottedLine
\RightLabel{def}
\UnaryInfC{S_{v+1}}
\RightLabel{TI}
\UnaryInfC{$T(\ulcorner S_{v+1} \urcorner)$}
\RightLabel{$\to E$}
\BinaryInfC{$\bot$}
\end{prooftree}
$$

$\Delta_1(v)$ has maximum formulas, $T(\ulcorner S_v \urcorner)$ and $\forall x(x > v \to \neg T(\ulcorner S_x \urcorner))$ The application of the reduction procedure $\rhd_{T(x)}$ produces the above derivation $\Delta_2(v)$. $\Delta_2(v)$ also has a maximum formula, $\forall x(x > v \to \neg T(\ulcorner S_x \urcorner))$, and so is reduced to the following derivation $\Delta_3(v+1)$

by $\rhd_{\forall}$ and $\rhd_{\rightarrow(\emptyset)}$.

$$
\begin{prooftree}
\AxiomC{$[T(\ulcorner S_{v+1} \urcorner)]^1$}
\noLine
\UnaryInfC{$\theta(v+1)$}
\noLine
\UnaryInfC{$\bot$}
\RightLabel{$\rightarrow I_{,1}$}
\UnaryInfC{$\neg T(\ulcorner S_{v+1} \urcorner)$}
\AxiomC{$[T(\ulcorner S_w \urcorner)]^2$}
\noLine
\UnaryInfC{$\theta(w)$}
\noLine
\UnaryInfC{$\bot$}
\RightLabel{$\rightarrow I_{,2}$}
\UnaryInfC{$\neg T(\ulcorner S_w \urcorner)$}
\RightLabel{$\rightarrow I_{,\emptyset}$}
\UnaryInfC{$w > v+1 \rightarrow \neg T(\ulcorner S_w \urcorner)$}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall x(x > v+1 \rightarrow \neg T(\ulcorner S_x \urcorner))$}
\dottedLine
\RightLabel{def}
\UnaryInfC{S_{v+1}}
\RightLabel{TI}
\UnaryInfC{$T(\ulcorner S_{v+1} \urcorner)$}
\RightLabel{$\rightarrow E$}
\BinaryInfC{$\bot$}
\end{prooftree}
$$

However, in $\Delta_3(v+1)$, the occurrence of $\neg T(\ulcorner S_{n+1} \urcorner)$ is a maximum formula. The application of $\rhd_{\rightarrow}$ to $\Delta_3(v+1)$ yields the following derivation $\Delta_4(v+1)$.

$$
\begin{prooftree}
\AxiomC{$[T(\ulcorner S_w \urcorner)]^1$}
\noLine
\UnaryInfC{$\theta(w)$}
\noLine
\UnaryInfC{$\bot$}
\RightLabel{$\rightarrow I_{,1}$}
\UnaryInfC{$\neg T(\ulcorner S_w \urcorner)$}
\RightLabel{$\rightarrow I_{,\emptyset}$}
\UnaryInfC{$w > v+1 \rightarrow \neg T(\ulcorner S_w \urcorner)$}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall x(x > v+1 \rightarrow \neg T(\ulcorner S_x \urcorner))$}
\dottedLine
\RightLabel{def}
\UnaryInfC{S_{v+1}}
\RightLabel{TI}
\UnaryInfC{$T(\ulcorner S_{v+1} \urcorner)$}
\noLine
\UnaryInfC{$\theta(v+1)$}
\noLine
\UnaryInfC{$\bot$}
\end{prooftree}
$$

$\Delta_4(v+1)$ has the same form of $\Delta_1(v)$ except its variable. The same reduction procedures to $\Delta_4(v+1)$ will yield the variant $\Delta_{3i+1}(v+i)$ of $\Delta_1(v)$ where $0 \leq i$. Hence, $\Delta_1(v)$ initiates a non-terminating reduction sequence and so is not normalizable. $\square$

Proposition 2.4 uses the id est inferences from S_v to $\forall x(x > v \rightarrow \neg T(\ulcorner S_x \urcorner))$ and vice versa. The derivation $\Delta_1(v)$ satisfies *TCP*. Although both Σ_3 of Proposition 2.3 and $\Delta_1(v)$ of Proposition 2.4 generate non-terminating reduction sequences, Tennant (1995) calls the former a *looping* reduction sequence and the latter a spiraling reduction sequence. Since he believed that Yablo's paradox lacks the self-referential expression but the Liar paradox has, Tennant (1995, p. 207) claimed,

> I shall make so bold as to suggest that it is precisely when the non-terminating reduction procedures enter loops that self-reference is involved. And when they don't enter loops - as with Yablo's example - then self-reference is not involved.

He thought that any derivations of paradoxes satisfying *TCP* (i.e. T-paradoxes) generate a looping reduction sequence if the self-reference is involved in them; otherwise, they produce a different type of infinite reduction sequence, such as a spiraling reduction sequence. His claim can be regarded as the following informal conjecture for self-referential paradoxes.[6]

Tennant's Conjecture for Self-Referential Paradoxes(*TCSP*): Let $\mathfrak{D}$ be any derivation satisfying *TCP*, i.e. a T-paradox. $\mathfrak{D}$ generates a looping reduction sequence if and only if $\mathfrak{D}$ formalizes a self-referential paradox.

Although Tennant (1995) made so bold as to suggest *TCSP*, *TCSP*

[6] Since *TCSP* contains informal expressions such as a 'self-referential paradox,' 'a looping reduction sequence,' and 'formalize,' we will consider it to be an informal conjecture.

has problems that it should be established. In the next section, we will consider two minor problems and a counterexample to *TCSP*.

3 Problems of Tennant's Conjecture for Self-Referential Paradoxes(*TCSP*)

One of the minor problems of *TCSP* is that it should be clearly established that Yablo's paradox is a paradox without self-reference. Unfortunately, whether Yablo's lacks the self-reference is still arguable. Although Tennant (1995) assumed that it is a liar-like paradox without self-reference, Graham Priest (1997, p. 238) argued that Yablo's paradox has 'a fixed point ... of exactly the same self-referential kind as in the liar paradox.' Soon after, Roy Sorensen (1998) defended the non-circularity of Yablo's paradox, but Jc Beall (2001) sided with Priest against Yablo and Sorensen. If Yablo's paradox were the same self-referential kind and a formalization of it generates a spiraling reduction sequence, *TCSP* would be false.

Even if it is established that Yablo's paradox is a non-self-referential paradox, it can be asked whether Tennant's spiraling reduction sequence is different from his looping reduction sequence.

$$\begin{array}{c}
[T(\ulcorner S_w \urcorner)]^1 \\
\theta(w) \\
\dfrac{\dfrac{\dfrac{\bot}{\neg T(\ulcorner S_w \urcorner)} \to I_{,1}}{w > v \to \neg T(\ulcorner S_w \urcorner)} \to I_{,\emptyset}}{\forall x(x > v \to \neg T(\ulcorner S_x \urcorner))} \forall I \\
\cdots\cdots\cdots\cdots\ def \\
\dfrac{S_v}{T(\ulcorner S_v \urcorner)} TI \\
\theta(v) \\
\bot
\end{array}
\qquad
\begin{array}{c}
[T(\ulcorner S_w \urcorner)]^1 \\
\theta(w) \\
\dfrac{\dfrac{\dfrac{\bot}{\neg T(\ulcorner S_w \urcorner)} \to I_{,1}}{w > v+1 \to \neg T(\ulcorner S_w \urcorner)} \to I_{,\emptyset}}{\forall x(x > v+1 \to \neg T(\ulcorner S_x \urcorner))} \forall I \\
\cdots\cdots\cdots\cdots\ def \\
\dfrac{S_{v+1}}{T(\ulcorner S_{v+1} \urcorner)} TI \\
\theta(v+1) \\
\bot
\end{array}$$

The derivation $\Delta_1(v)$ of Proposition 2.4 generates a non-terminating reduction sequence through the derivation $\Delta_4(v+1)$. The left derivation above is $\Delta_1(v)$ and the right derivation above is $\Delta_4(v+1)$. The main difference between them is the numerical index of variables but not the form of each derivation in the reduction sequence. If the proof-theoretic analysis of paradoxes focusses on the form of paradoxical derivation, it should be explicated how different two derivations are in the form of derivations. With respect to the form of derivations, it could be regarded as a looping reduction sequence because $\Delta_1(v)$ and $\Delta_4(v+1)$ share the same form except the index number of variables. If Tennant's spiraling reduction sequence were not distinguished from a looping reduction, *TCSP* could be a false conjecture.

The other problem of *TCSP* is that it has a counterexample. We assume for clarity that the Liar paradox is a self-referential paradox, while Yablo's is not. Also, it will be assumed that a spiraling reduction sequence is clearly distinct from a reduction loop. Then, we will show that there is a derivation formalizing the Liar paradox which generates a spiraling reduction sequence. Since the Liar paradox is a self-referential paradox, the derivation shows that *TCSP* is false.

Let S_{CL} be a natural deduction system containing rules for $\rightarrow$ $, T(x)$, the liar sentence Φ, and $CR-$rule. S_{CL} has the set $\mathbb{R}_{CL}$ of reduction procedures including reductions for $\rightarrow, T(x), \Phi$, and the following reduction for $CR-$rule.

$$\dfrac{\dfrac{\begin{matrix}[\neg\varphi]^1\\ \mathfrak{D}_1\\ \bot\end{matrix}}{\varphi}\,CR_{,1} \qquad \begin{matrix}\mathfrak{D}_2\\ \psi\end{matrix}}{\sigma}\circ E \qquad \rhd_{CR(\circ)} \qquad \dfrac{\begin{matrix}\dfrac{\dfrac{[\neg\sigma]^2 \qquad \dfrac{[\varphi]^1 \qquad \begin{matrix}\mathfrak{D}_2\\ \psi\end{matrix}}{\sigma}\circ E}{\bot}\to E}{\neg\varphi}\to I_{,1}\\ \mathfrak{D}_1\\ \bot\end{matrix}}{\sigma}\,CR_{,2}$$

$\rhd_{CR(\circ)}$ is an instance of the reduction procedure for $CR-$rule introduced by Gunnar Stålmarck (1991, pp. 131-132). Stålmarck used the reduction when he proposed a proof for normalization of (full) classical natural deduction system. Then, we have the following result.

Proposition 3.1. *There is a closed derivation of* $\bot$ *in* S_{CL} *with respect to* $\mathbb{R}_{CL}$*, which generates a non-terminating reduction sequence and so is irreducible.*

Proof. Two claims will verify the result.
Claim 1. There is a closed derivation Π_3 of $\bot$.

There is an open derivation Π_1 of $\bot$ from $[\neg\Phi]$ as left below. With Π_1, there is an open derivation Π_2 of $\bot$ from $[\neg T(\ulcorner\Phi\urcorner)]$ as right below.

$$\dfrac{[\neg\Phi]^2 \qquad \dfrac{\dfrac{[\neg\Phi]^2 \qquad \dfrac{[T(\ulcorner\Phi\urcorner)]^1}{\Phi}\,TE}{\bot}\to E}{\Phi}\,\Phi I_{,1}}{\bot}\to E \qquad\qquad \dfrac{[\neg T(\ulcorner\Phi\urcorner)]^3 \qquad \dfrac{\dfrac{\begin{matrix}[\neg\Phi]^2\\ \Pi_1\\ \bot\end{matrix}}{\Phi}\,CR_{,2}}{T(\ulcorner\Phi\urcorner)}\,TI}{\bot}\to E$$

Having Π_1 and Π_2, there is a closed derivation Π_3 of $\bot$.

$$\dfrac{\dfrac{\begin{matrix}[\neg\Phi]^4\\ \Pi_1\\ \bot\end{matrix}}{\Phi}\,CR_{,4} \qquad \begin{matrix}[\neg T(\ulcorner\Phi\urcorner)]^3\\ \Pi_2\\ \bot\end{matrix}}{\bot}\,\Phi E_{,3}$$

Claim 2. Π_3 generates a non-terminating reduction sequence and so is irreducible with respect to $\mathbb{R}_{CL}$.

Π_3 is reducible to the following derivation Π_4 by $\rhd_{CR(\Phi)}$.

$$\dfrac{\begin{matrix}\dfrac{\dfrac{[\neg\bot]^5 \qquad \dfrac{[\Phi]^6 \qquad \begin{matrix}[\neg T(\ulcorner\Phi\urcorner)]^3\\ \Pi_2\\ \bot\end{matrix}}{\bot}\,\Phi E_{,3}}{\bot}\,\to E}{\neg\Phi}\,\to I_{,6}\\ \Pi_1\\ \bot\end{matrix}}{\bot}\,CR_{,5}$$

Π_4 is restated as below.

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg\bot]^5 \qquad \dfrac{[\Phi]^7 \qquad \begin{matrix}[\neg T(\ulcorner\Phi\urcorner)]^8\\ \Pi_2\\ \bot\end{matrix}}{\bot}\,\Phi E_{,8}}{\bot}\,\to E}{\neg\Phi}\,\to I_{,7} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\neg\bot]^5 \qquad \dfrac{[\Phi]^6 \qquad \begin{matrix}[\neg T(\ulcorner\Phi\urcorner)]^3\\ \Pi_2\\ \bot\end{matrix}}{\bot}\,\Phi E_{,3}}{\bot}\,\to E}{\neg\Phi}\,\to I_{,6} \qquad \dfrac{[T(\ulcorner\Phi\urcorner)]^1}{\Phi}\,TE}{\bot}\,\to E}{\Phi}\,\Phi I_{,1}}{\bot}\,\to E}{\bot}\,CR_{,5}$$

Since Π_4 has a maximum formula $\neg\Phi$, we apply $\rhd_{\rightarrow}$ to Π_4 twice and then the derivation has a maximum formula Φ. Then, by applying $\rhd_{\Phi}$ twice, we have the derivation Π_5 below.

$$
\dfrac{\dfrac{[\neg\bot]^5 \qquad \dfrac{\dfrac{\dfrac{[\neg\bot]^5 \qquad \dfrac{\dfrac{[T(\ulcorner\Phi\urcorner)]^1}{\Phi}\,TE \qquad \begin{array}{c}[\neg T(\ulcorner\Phi\urcorner)]^3\\ \Pi_2\\ \bot\end{array}}{\bot}\,\Phi E_{,3}}{\bot}\rightarrow E}{\neg T(\ulcorner\Phi\urcorner)}\rightarrow I_{,1} \qquad \dfrac{\dfrac{\begin{array}{c}[\neg\Phi]^2\\ \Pi_1\\ \bot\end{array}}{\Phi}\,CR_{,2}}{T(\ulcorner\Phi\urcorner)}\,TI}{\bot}\rightarrow E}{\bot}\rightarrow E}{\bot}\,CR_{,5}
$$

To eliminate the maximum formulas $\neg T(\ulcorner\Phi\urcorner)$ and $T(\ulcorner\Phi\urcorner)$, we apply $\rhd_{\rightarrow}$ and $\rhd_{T(x)}$ in order. Then, we have the derivation Π_6 below.

$$
\dfrac{\dfrac{[\neg\bot]^5 \qquad \dfrac{[\neg\bot]^5 \qquad \dfrac{\dfrac{\begin{array}{c}[\neg\Phi]^2\\ \Pi_1\\ \bot\end{array}}{\Phi}\,CR_{,2} \qquad \begin{array}{c}[\neg T(\ulcorner\Phi\urcorner)]^3\\ \Pi_2\\ \bot\end{array}}{\bot}\,\Phi E_{,3}}{\bot}\rightarrow E}{\bot}\rightarrow E}{\bot}\,CR_{,5}
$$

Π_6 includes the same derivaiton with Π_3. Again Π_3 can be further reduced. Then, we have the following infinite reduction sequence.

$$\dfrac{\dfrac{[\neg\bot]^5 \qquad \dfrac{[\neg\bot]^5 \qquad \dfrac{\dfrac{[\neg\bot]^9 \qquad \dfrac{[\neg\bot]^9 \qquad \begin{matrix} \dfrac{\dfrac{[\neg\bot]^i \qquad \dfrac{[\neg\bot]^i \quad \begin{matrix}\vdots \\ \bot\end{matrix}}{\bot}\to E}{\bot}\to E}{\bot}CR_{,i} \\ \vdots \\ \bot \end{matrix}}{\bot}\to E}{\bot}\to E}{\bot}CR_{,9}}{\bot}\to E}{\bot}\to E}{\bot}CR_{,5}$$

where $i = 4j+1 (j > 0)$. Therefore, Π_3 initiates a non-terminating reduction sequence and is irreducible with respect to $\mathbb{R}_{CL}$. $\square$

The derivation Π_3 initiates a non-terminating reduction sequence which is not so much a looping reduction as a spiraling reduction. The problem is that Π_3 formalizes the Liar paradox. Since the Liar paradox is a self-referential paradox, Proposition 3.1 shows that *TCSP* is false.

Tennant may answer that Proposition 3.1 is not a real counterexample to *TCSP*, in that Tennant (2017, pp. 281–285) thought that the law of excluded middle has no essential role in the derivation of the paradoxes. The law of excluded middle provably implies *classical reductio*. It can be said that he has the same stance on *classical reductio*. He probably thought that any correct formalization of paradoxes in natural deduction must not use $CR-$rule because $CR-$rule is redundant to formalize paradoxes. However, in the next section, we shall argue that his use of $\Phi I-$ and $\Phi E-$rules has the same role of $CR-$rule with respect to the Liar sentence Φ. In this sense, it might be too hasty to reject the use of $CR-$rule in any formalization of paradoxes.

4 Shouldn't We Use $CR-$Rule in Paradoxical Reasoning?

Tennant (2017, pp. 281–285) claimed that the law of excluded middle, which implies *classical reductio*, has no essential role in the derivation of paradoxes. Furthermore, Tennant (2015, p. 589) thought that the use of *classical reductio* has masked the main feature of paradoxical reasoning, such as the non-terminating reduction sequence, and proposed the methodological conjecture that paradoxes are never strictly classical.[7] Both claims are based on his view that there is no paradox whose associated derivations of $\perp$ have to make use of $CR-$rule. If he were correct, Proposition 3.1 cannot be the proper counterexample to *TCSP*.

However, the situation changes if we notice that the role of *classical reductio* is melted in the rules of the system formalizing Tennant's derivation of the Liar paradox. For instance, $\perp$ in the premises of $CR-$rule disappears in the conclusion after applying $CR-$rule. Likewise, the use of $\Phi I-$rule disguises $\perp$ in its premises and derives a single liar sentence Φ. When Tennant (2016, 2017) claimed that the Liar paradox is a genuine paradox, he used a derivation similar to Σ_3

[7] For the discussion of the problems of the methodological conjecture, the reader can consult Choi (2019, Ch. 9).

of Proposition 2.3. $\Phi-$rules play an essential role of his derivation of the Liar paradox. We only apply $\Phi-$rules to derivation concerning Φ. If the use of rules has the same consequence of $CR-$rule with respect to Φ, he should admit that $CR-$rule has an essential role of paradoxical reasoning and looses the main reason why $CR-$rule must not be used in paradoxical reasoning. Then, Proposition 3.1 can be the counterexample to *TCSP*.

As we have seen in Proposition 2.3, the Liar paradox is formalized in S_L with the applications of $\Phi I-$ and $\Phi E-$rules. Since his derivation of the Liar paradox does not use $CR-$rule, he considers $CR-$rule has no essential role to formalize the Liar paradox in natural deduction. Unfortunately, he overlooks the possibility that the rules in S_L have the same consequence of $CR-$rule with respect to Φ. It can be claimed that the rules in S_L plays the same role of *classical reductio*.

While investigating the formalization of the Liar-type paradoxes, Choi (2018) has shown that, for a liar sentence Φ, if $\Phi \leftrightarrow \neg T(\ulcorner \Phi \urcorner)$ is assumed, then $\neg\neg\Phi \rightarrow \Phi$ is provable in natural deduction. The result represents that the structure of the derivation of the Liar paradox admits the inference from $\neg\neg\Phi$ to Φ, which is often regarded as the result of *classical reductio*. Similarly, by applying rules in S_L, $\neg\neg\Phi \rightarrow \Phi$ is derivable in S_L. Below in Proposition 4.1, although S_L does not have $CR-$rule, for a liar sentence Φ, S_L has the same consequence of $CR-$rule.

Proposition 4.1. *For a liar sentence Φ, there is a closed derivation of $\neg\neg\Phi \rightarrow \Phi$ in S_L.*

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[T(\ulcorner\Phi\urcorner)]^5}{\Phi}\,TI \qquad \dfrac{\dfrac{\dfrac{[\neg\neg\Phi]^1 \qquad \dfrac{\dfrac{[\neg T(\ulcorner\Phi\urcorner)]^2 \quad \dfrac{[\Phi]^3}{T(\ulcorner\Phi\urcorner)}\,TI}{\bot}\to E}{\neg\Phi}\to I_{,3}}{\bot}\to E}{\neg\neg T(\ulcorner\Phi\urcorner)}\to I_{,2} \qquad [\neg T(\ulcorner\Phi\urcorner)]^4}{\bot}\to E}{\bot}\,\Phi E_{,4}}{\Phi}\,\Phi I_{,5}}{\neg\neg\Phi\to\Phi}\to I_{,1}}{}$$

□

As we can see the left derivation below, $CR-$rule derives Φ from the derivation of $\bot$ from $[\neg\Phi]$. The right derivation below shows that the same result is derivable through Proposition 4.1.

$$\dfrac{\begin{matrix}[\neg\Phi]^1\\ \mathfrak{D}\\ \bot\end{matrix}}{\Phi}\,CR_{,1} \qquad\qquad \dfrac{\dfrac{}{\neg\neg\Phi\to\Phi}\,Prop.4.1. \quad \dfrac{\begin{matrix}[\neg\Phi]^1\\ \mathfrak{D}\\ \bot\end{matrix}}{\neg\neg\Phi}\to I_{,1}}{\Phi}\to E$$

It is not to say that $CR-$rule is admissible in S_L.[8] One may claim the above result is limited to a particular sentence, such as the liar sentence Φ. $CR-$rule applies to any formula but the result presented is established only for Φ. For the proof of admissibility of $CR-$rule in S_L, we may use the following rules which has the same form of

[8]For a given natural deduction system S, a rule with the premises $\varphi_1,...,\varphi_n$ and the conclusion ψ is said to be *admissible* for S if, whenever an instance of $\varphi_1,...,\varphi_n$ is derivable in S, the corresponding instance of ψ is derivable in S.

Φ-rules but can apply to any formula.

$$\dfrac{\begin{matrix}[T(\ulcorner\varphi\urcorner)]^1\\ \mathfrak{D}\\ \bot\end{matrix}}{\varphi}\,\varphi I_{,1} \qquad \dfrac{\varphi \quad \begin{matrix}[\neg T(\ulcorner\varphi\urcorner)]^1\\ \mathfrak{D}\\ \psi\end{matrix}}{\psi}\,\varphi E_{,1}$$

Plus, our focus is only on paradoxical derivations, also on the claim that *classical reductio* should not be used in any formalization of paradoxes. Since the derivation of the Liar paradox only concerns Φ, it is sufficient to show that S_L has the same consequence of $CR-$rule with respect to Φ.

In sum, it is too hasty to claim that *classical reductio* has no essential role in the paradoxical derivations. If Tennant claims that $CR-$rule must not be used because of its essential role in paradoxical reasoning, Proposition 4.1 should not be proved in S_L. If $\Phi I-$ and $\Phi E-$rules have the essential role in paradoxical reasoning, since, for a liar sentence Φ, $\Phi-$rules share the role of classical inference from $\neg\neg\Phi$ to Φ, $CR-$rule with respect to Φ can have the same essential role in paradoxical reasoning. Therefore, it is unconvincing that any counterexample applying $CR-$rule is not a legitimate counterexample to *TCSP*.

5 Conclusion

So far, we have investigated Tennant's conjecture for self-referential paradoxes in Section 2.3 that for any T-paradox $\mathfrak{D}$, $\mathfrak{D}$ generates a looping reduction sequence if and only if $\mathfrak{D}$ formalizes a self-referential paradox (*TCSP*). First, Tennant's spiraling reduction, i.e. the reduction processes used in Proposition 2.4, has the problem of iterating the same form of reduction procedures except the numerical index of variables. Focussing on the form of reduction procedures, it is hard to say that the reduction procedures has the spiral form. Rather, the reduction procedure suggested in Proposition 3.1 can be separated from looping reduction sequences and can be called a spiral reduction. However, the derivation Π_3 which generates a spiral reduction formalizes the Liar paradox. Since the Liar paradox is a self-referential paradox, Proposition 3.1 can be a counterexample to *TCSP*.

Proposition 3.1 applies the rule of *classical reductio*, $CR-$rule. Tennant may not think that Proposition 3.1 is the counterexample to *TCSP* because he believes that *classical reductio* has no essential role to formalize the Liar paradox. To answer against the plausible rebuttal, Section 4 argues that, for a liar sentence Φ, S_L derivaes $\neg\neg\Phi \rightarrow \Phi$ without the application of $CR-$rule. So to speak, the rules in S_L which formalizing the Liar paradox already shares the role of *classical reductio* as far as Φ is concerned. It is not yet established that *classical reductio* has no essential role to formalize the Liar paradox. Therefore, Proposition 3.1 is the counterexample to *TCSP*.[9]

The spiraling reduction procedure in Proposition 3.1 is raised because of the reduction process $\triangleright_{CR(\circ)}$ introduced by Stålmarck (1991). If Proposition 3.1 is not a proper counterexample to *TCSP*, it is because the reduction process $\triangleright_{CR(\circ)}$ is not an admissible reduction procedure for S_L. However, Tennant has never concerned with the criterion for admissible reduction procedures. On the other hand, Choi (2019, Ch. 3) and Schroeder-Heister and Tranchini (2017, Sec. 5) have proposed their criteria of admissible reduction procedures.

[9] An anonymous reviewer noticed that it would be better to present a counterexample in which classical rules were not used. To find a counterexample without classical rules is left for the further research.

The questions of whether their criteria are legitimate and whether $\triangleright_{CR(\circ)}$ becomes an admissible reduction process in accordance with their criteria require further research.

References

Beall, Jc. (2001), "Is Yablo's paradox non-circular?", *Analysis*, 61(3), pp. 176–187.

Choi, S. (2018), "Liar-Type Paradoxes and Intuitionistic Natural Deduction Systems", *Korean Journal of Logic*, 21(1), pp. 59–96.

Choi, S. (2019), *On Proof-Theoretic Approaches to the Paradoxes: Problems of Undergeneration and Overgeneration in the Prawitz-Tennant Analysis*, Doctoral dissertation, Korea Univeristy.

Ekman, J. (1998), "Propositions in propositional logic provable only by indirect proof", *Mathematical Logic Quarterly*, 44(1), pp. 69–91.

Gentzen, G. (1935), "Investigations concerning logical deduction", In M. E. Szabo (Eds.), *The Collected Papers of Gerhard Gentzen*, Amsterdam and London:North-Holland, pp. 68–131.

Prawitz, D. (1965), *Natural Deduction: A Proof-Theoretical Study*, Dover Publications.

Prawitz, D. (1971), "Ideas and results in proof theory", In J. Fenstad (Ed), *Proceedings of the 2nd Scandinavian Logic Symposium*, Amsterdam: North Holland, pp. 235-308.

Priest, G. (1997), "Yablo's paradox", *Analysis*, 57(4), pp. 236–242.

Schroeder-Heister, P. and Tranchini, L. (2017), "Ekman's paradox", *Notre Dame Journal of Formal Logic*, 58(4), pp. 567–581.

Schroeder-Heister, P. and Tranchini, L. (2018), "How to Ekman a Crabbé-Tennant", *Synthese*, `https://doi.org/10.1007/s11229-018-02018-3`.

Stålmark, G. (1991), "Normalization Theorems for Full First Order Classical Natural Deduction", *The Journal of Symbolic Logic*. 56(1), pp. 129-149.

Sorensen, R. (1998), "Yablo's paradox and Kindred infinite Liars", *Mind*. 107(425), pp. 137–155.

Tennant, N. (1982), "Proof and Paradox", *Dialectica*, 36, pp. 265–296.

Tennant, N. (1995), "On paradox without self-reference", *Analysis*, 55(3), pp. 199–207.

Tennant, N. (2015), "A new unified account of truth and paradox", *Mind*, 124, pp. 571–605.

Tennant, N. (2016), "Normalizability, cut eliminability and paradox", *Synthese*, https://doi.org/10.1007/s11229-016-1119-8.

Tennant, N. (2017), *Core Logic*, Oxford University Press.

van Dalen, D. (2013), *Logic and Structure (5th ed.)*, London: Springer-Verlag Press.

von Plato, J. (2013), *Elements of Logical Reasoning*, Cambridge University Press.

Yablo, S. (1993), "Paradox without self-reference", *Analysis*, 53(4), pp. 251–252.

고려대학교 철학과 강사
Department of Philosophy, Korea University
choi.seungrak.eddy@gmail.com

테넌트의 자기지시적 역설에 관한 가설과 그에 대한 고전적 반례

최 승 락

니일 테넌트는 그의 “자기지시성 없는 역설”에서 자기지시적 역설에 관한 증명론적 가설을 제시한다. 이는 자기지시적 역설을 자연연역에서 형식화한 도출은 모두 항상 고리형 환원열을 양산한다는 것이다. 그에 따르면, 거짓말쟁이 역설을 형식화한 도출은 고리형 환원열을 양산하며 자기지시적 역설이 아닌 야블로의 역설을 형식화한 도출은 꽈베기형 환원열을 양산한다.

이 글에서 필자는 거짓말쟁이 역설을 형식화한 자연연역의 도출도 꽈베기형 환원열을 양산할 수 있음을 보일 것이다. 거짓말쟁이 역설은 대표적인 자기지시적 역설이기 때문에 이 결과는 테넌트의 가설에 대한 반례가 될 것이다.

마지막으로 테넌트는 고전적 귀류법이 역설을 형식화하는데 실질적인 역할을 못 한다고 생각하는데 이런 측면에서 그는 제시된 반례를 받아들이지 않을 수 있다. 하지만 필자는 고전적 귀류법과 그의 거짓말쟁이 문장에 관한 규칙이 어떤 추론적 역할을 공유함을 논할 것이며 역설을 형식화하는데 고전적 귀류법을 사용하지 않아야 한다면 그의 규칙도 사용하지 않아야 함을 논할 것이다.

주요어: 거짓말쟁이 역설, 야블로의 역설, 자기지시성, 고전적 귀류법, 구나 스톨마크, 니일 테넌트.